ANALYSIS OF SOME STOCHASTIC MODELS AND ITS APPLICATIONS

A THESIS SUBMITTED TO BUNDELKHAND UNIVERSITY
FOR THE DEGREE
OF
DOCTOR OF PHILOSOPHY

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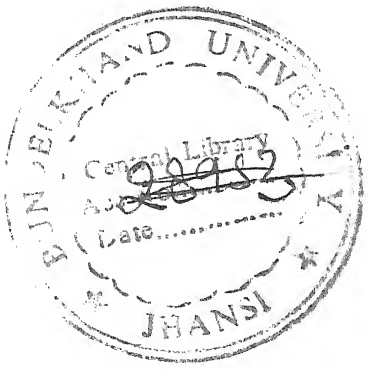

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DECLARATION

This thesis entitled " **ANALYSIS OF SOME STOCHASTIC MODELS AND ITS APPLICATIONS** " that is being submitted by me at the Bundelkhand University , Jhansi, U. Pradesh for award of the degree of Doctor of Philosophy is based on my research work carried out under the Supervis' ion and guidance of Dr. **Vijay Kumar Sehgal** , Reader, Department of Mathematics & Statistics , Bundelkhand University. This work has not been submitted to any University or Institution for the award of any degree.

V.Sehgal 28/5/98

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Batra

(Deepak Batra)

CERTIFICATE

This is to certify that the work embodied in the thesis "Analysis of some stochastic models and its applications" by Deepak Batra, for the award of the Degree of Doctor of Philosophy is a record of bonafide research work carried out by him under my supervision and guidance and has not been submitted elsewhere for a Degree/Diploma in any form.

It is further certified that he has worked with me for the period required under clause seven of the Bundelkhand University ordinance.

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(Deepak Batra)

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Application of Stochastic models in fertility has been an important component of investigation in demography concerning the human reproductive process. Some researchers have shown their interest to develop probability models for the number of conceptions in a certain period giving rise to discrete probability models (Singh 1963), (Pathak 1966). A good amount of research has been carried out related to probability models of the waiting time distribution for the first conception as well as between conceptions of different orders.

The researchers involved in the field of demography have utilized opportunity to develop analytic model building utilizing the theory of Stochastic Processes and Renewal Theory. Family building models are particularly concerned with the series of conceptions, births etc. that constitute the reproductive history of a couple.

Another aspect of this type of research in this area is to find practical solutions to problems connected with decision making and evaluation of family planning programmes. The problem of measuring the impact of family planning programme on fertility has become a major research issue for demographers as well as policy makers. Mathematical demographers have contributed considerable literature regarding fertility analysis with Stochastic models. These models have advantage over the usual deterministic models. For example, with stochastic models in fertility analysis one can obtain sample estimates of the demographic parameters with exact specification of Sampling errors. Considerable researchers have worked out the probability of conception allowing for the non-fecund period of pregnancy. Models have generated to estimate these probabilities (some researchers have taking foetal wastage into consideration on the other hand some have ignored by assuming complete pregnancy leading to live birth). These research work are based on the data source providing the distribution of birth intervals between two successive birth orders. Mitra and Bannerjee (1982) while estimating the probability distribution between two successive live births also

investigated the same distributions in a different way to estimate the lengths of the non susceptible period that begin from the time to conception.

Since conception can occur during only a small segment of the ovrian cycle and mean length of the cycle is about 30 days thus the conception time was treated as i) Discrete (each unit being taken equal to one month)

ii) Continuous.

The variables under consideration in such probability models are thus the random lengths of intervals between events. The family building models were thus introduced in both discrete and continuous time considering both hetrogeneous and homogeneous population. The importance of stochastic models in fertility analysis has been encountered by Talwar (1966). By considering the distribution of the time interval between consecutive conceptions he has obtained the joint distribution of the period of post partum amenorrhea, gestation period and fecundable period. From this model one can estimate the parameters concerning the period of post-partum amenorrhoea

(P.P.A.) from the data of interlive birth intervals which are easily available from most of the retrospective demography surveys.

Biswas (1975) has shown that with application of stochastic models in fertility analysis one can estimate proportion of women exposed to the risk of fecundity, the proportion under family planning practice and from which the proportion of births interrupted due to family planning practice can also be estimated. Stochastic models can also be used in analysing demographic data based on follow up surveys. In such surveys either the period within which data is collected is taken as fixed so that the proportion of observations under consideration during that period is a discrete or continuous random variable within which a fixed number of observations under consideration lie. There had been a study with a new investigation which may highlight the interrelationship between two useful variables in the human reproduction process; viz the period of post partum amenorrhoea and the lactation period which are useful items of consideration for averting methods which has been done by the application of a bivariate survival model due to Freund (1961).

1.2 REVIEW OF EARLIER STOCHASTIC FERTILITY MODELS:

The application of stochastic models in human fertility started in the year 1924 with C. Gini's paper presented at the International Mathematical Congress in Toronto. In recent times considerable interest is generated in the analysis of pregnancies of cohorts of married women. Henry (1953) has emphasised on the patterns of live births. The human reproductive process is thus compared with Geiger Muller Counter where every event occurring with an intensity is followed by a dead time (or blocked time) during which no further success is possible. G. M. counter is classified according as the dead time is fixed or random variable. The term "counter is paralyzed" is used in this respect which is comparable to the non - susceptible period following a pregnancy in the human reproductive process. The type II G. M. counter are contrasted from type I in the sense that not only there cannot be any success during the dead time in type II during the dead time further prolongs the dead time without registering the event during the blocked period.

For the analysis of human reproductive process, counter model I with fixed or random variable dead time appears to be more appropriate. Human reproductive process are directly derivable from the existing results of G. M. counter models. Further the renewal theory technique is amenable for the development of probability models on fertility and the human reproductive process. The models can be classified in two categories:

1. Deals with the waiting time distribution between two consecutive events.
2. Deals with the probability distribution of the number of events during a fixed time.

In first category time is a random variable and the system conforms to ordinary or delayed Renewal process (Non - Markovian in character). In the Second category the number of events is a random variable during fixed time and conforms to a renewal counting process.

In India, the making of Stochastic model representing the human reproductive process was first done by Dandekar (1955). Where Dandekar assumed one to

one correspondence between pregnancy and time birth and a constant time independent Poisson intensity for births/ pregnancies of all others. A fixed non-susceptible period (dead time) was assumed following every pregnancy/ birth in the pattern of counter Model type I. The probability distribution of the number of events in fixed marital duration (o,t] has been obtained from the Modified Poisson Distribution as evolved by Dandekar. The distribution considered is Poisson Renewal Process where every renewal is followed by a dead time of fixed measure. Thus it is obtainable as an example of Counter Model Type I with fixed dead time as considered by Takacs (1960), Dharmadhikari (1964), Biswas (1973). Basu (1955) adopted difference equation techniques which provides alternative derivation of asymptomatic pregnancy rate.

Brass (1958) obtained Negative model from Dandekar's distribution ignoring the non-susceptible exposures following every birth and obtained a theoretical model for the probability distribution of births. However, the model did not give good fit in empirical data as infecundable exposure following every birth was ignored, the probability of zero number of

births to mothers with completed reproductive span as obtainable from William Brass Model did not show good correspondence with that of the empirical frequency for the same class because of presence of considerable proportion of Biologically sterile mothers(or couples) as well as a good proportion of mothers (or couples) who wanted to remain effectively childless without being biologically sterile.

Singh (1963) independently obtained two different interrupted probability models on similar assumption with the problem of estimating the number of schools of fishes in a fishing ground. For example, the dead time "h" following each conception was comparable to the time for which further counting of fishes was temporarily abandoned following the sight of a school of fish and exactly "h " hours were spent for fishing the school. In both the models Singh assumed the presence of $(i- \lambda)$ of the couples to be effectively sterile. While in the first model Singh assumed the monthly probability of conception to be constant, in the second model he assumed the same admitting a probability distribution of ß type following Potter and Parker.

Another aspect of S. N. Singh's work in this direction comprises of analysing the building up appropriate probability model for explaining the random changes causing variability in the number of conceptions during a fixed fecundable period. In the first course of his study Singh (1963) devoted to this analysis on complete conception only and obtained BAN estimates of the parameters of the model, whereas in a generalized model built up later, Singh and Bhatacharya (1970) admitted incomplete conceptions too in the model which led to several more parameters of the model. A partial estimation of some of the parameters was made while assuming the knowledge relating to the rest.

Dharmadhikari (1964) while attempting to generalize Dandekar's model , first of all developed a model based on Mixture of Exponential distribution with the origin shifted by K (the length of the non-susceptible exposure). Besides this generalization of Dandekar's model modified into a mixture of two distributions, Dharmadhikari developed a generalized time related dependent process (Doubly Stochastic Poisson Process) and considered the application of the same in further models.

S. N. Singh (1964) further obtained the interrupted probability models for couple fertility under the assumption of the presence of a fixed proportion of effectively sterile couple as well as fixed non- susceptible exposure following every conception and obtained BAN estimators of the parameters of the model by using Central Limit Theorem. Modified Minimum χ^2 method has been applied to estimate BAN estimators. As noted by Sheps and Menkin (1969), Singh's model is a modified restatement of Dandekar's model using a Neyman's Method (1949).

Perrin and Sheps (1964) made an explicit use of renewal Theory for building of probability models on human reproductive process that allow a variety of pregnancy outcomes (live birth, still birth, foetal wastage, spontaneous or induced abortions) and general distribution for durations of pregnancy and the post-partum non- susceptible status. The states of the reproductive process in Sheps and Perrin consist of:

S_0 = Non - pregnant fecundable state

S_1 = Pregnant state

S_2 = Post -partum infecundable state associated with abortion or early foetal loss

S_3 = Post -partum infecundable state associated with still birth

S_4 = Post -partum infecundable state associated with with live birth

The length of stay in each state is viewed as a random variable, the waiting between two renewals of the same state and the number of renewals of each state during a fixed term (o,t] conforms to a Semi Markov pattern. To study the fertility process it is important to know number of passages into a specified state , the fertility rate (or the pregnancy rate per unit of time); and the probability distribution of the renewal of different states of the process in a given period of time. The results obtained from these models have implication both for the analysis of data on the distribution of births to a group of women in a calender period and for the planning and evaluation of family planning programmes. In this direction a family building model has been set up by Sheps, Menkin and Radick (1969). A very important model for studying the

short term effects of change in reproductive behaviour (Sheps and Menkin (1971)) on intrinsic birth rate and a model for conceptive delays in a heterogeneous group have also come up.

Theoretical investigation for developing Inter-arrival time distribution under weighted Poisson Process by applying Palm Probability technique has been done. The renewal intervals corresponding to different order of conceptions become correlated because of variability between individuals thereby distributing the entire renewal structure. Multistage Markovian chain model with forward and backward transition probabilities represented by a density dependent birth and death process has been developed to study effectively the efficacy of a particular sterilization policy motivated to reduce the birth rate in a population. Attempts have been made to analyze the direct and indirect strategies for the reduction of conception rate. While classifying family limitations methods as direct and indirect , the inter-live birth intervals between 1-2, 2-3, & 3-4 have been worked out. G. M. counter technique has been applied to take account of the infecundable period. There has been a new application of survival analysis technique for

obtaining inter live birth intervals by employing a general hazard model which can take account of any type of fertility behaviour which one may theoretically assume (Pachal 1994).

1.3 A SURVEY OF SOME OF THE METHODOLOGIES EMPLOYED :

1. Palm probability is defined as the conditional probability of a specified number of events in a time interval given that an event has happened at the beginning of the interval. Cox and Isham (1980) describe Palm Probability as follows :

For $u > v$, let $N(u,v)$ be a random variable giving the number of events occurring in (u,v) and X_i be the sequence of the intervals between i th successive events $(i= 1,2,...)$ in a process starting from an arbitrary point.

Consider the survival function

$K_X(x) = P(X>x)$, where X is a r.v. representing the waiting time for first renewal

$$= \lim_{\delta \to 0+} P [N(0,x) = 0 | N(-\delta,0) > 0] ..... (1.01)$$

(subject to orderliness which implies not more than one renewal in the infinitesimal interval) which is the limiting probability that, given an event occurs immediately before the origin, the next event of the process occurs after that instant (i.e the system will survive more than a period x).

Now, by stationarity condition

$$P[N(0,x) = 0 \cap N(-\delta,0) > 0)]$$

$$= P[N(0,x) = 0] - P[N(-\delta,x)=0]$$

$$= P[N(x) = 0] - P[N(x+\delta)=0] \quad \ldots\ldots\ldots\ldots (1.02)$$

Now $P[N(0,x) = 0 \mid N(-\delta,0) > 0)]$

$$= \frac{P[N(0,x) = 0 \quad N(-\delta,0) > 0)]}{P[N(-\delta,0) > 0)]}$$

$$= \frac{P[N(x) = 0] - P[N(x+\delta) = 0]}{P[N(\delta) > 0)]}$$

$$\therefore P[N(0,x) = 0 \mid N(-\delta,0)>0]\ \delta^{-1}\ P[N((\delta>0]$$

$$= \delta^{-1}\ P[N(x)=0] - P[N(x+\delta)=0]]$$

$$= -\delta^{-1}\ [P[N(x+\delta)=0] - P[N(x)=0]] \quad \ldots\ldots\ldots\ldots (1.02')$$

Define

$$\lim_{\delta \to 0} \delta^{-1}\ P[N(\delta)>0] = \lambda \quad \ldots\ldots\ldots\ldots\ldots (1.03)$$

as the occurrence parameter λ of the process which we assume as finite.

Further, denoting

$$P[N(x) = k] = P_k(x), \quad k=0,1,2,\ldots\ldots\ldots\ldots \quad (1.04)$$

as the distribution of N(x), then in the limit as $\delta \to 0+$

$K_X(x) = P(X>x)$

$$= \lim_{\delta \to 0+} P[N(0,x) = 0 \mid N(-\delta,0)>0] \ldots\ldots\ldots \quad (1.04')$$

Now $P[N(0,x) = 0 \mid N(-\delta,0)>0]\ \delta^{-1}\ P[N(\delta)>0]$

$= K_X(x)\ \delta^{-1}\ P[N(\delta)>0]$ from (1.04') as $\delta \to 0+$

Since the process is orderly, if $k>0$, as $\delta \to 0+$

$p_k(x+\delta) = P[N(-\delta,x) = k]$

$= P[N(-\delta,0) = 0,\ N(0,x)=k] + P[N(-\delta,0)=1,\ N(0,x)=k-1] + O(\delta)$

$= p_k(x) - P[N(-\delta,0)>0,\ N(0,x)=k] + P[N(-\delta,0)>0,\ N(0,x)=k-1] + O(\delta)$

so that

$$p_k(x+\delta) - p_k(x) = - P[N(-\delta,0)>0,\ N(0,x)=k] + P[N(-\delta,0)>0,\ N(0,x)=k-1] + O(\delta)$$

$$= - P[N(-\delta,0)>0.\ P[N(0,x) = k|N(-\delta,0)>0] +$$

$$P[N(-\delta,0)>0.\ P[N(0,x)=k-1|N(-\delta,0)>0] + O(\delta)$$

Thus δ^{-1} $[P[N(-\delta,0)>o].\ P[N(0,x)=k|\ N(-\delta,0)>0]$

$$= - P[N(-\delta,0)>0].\ P[N(0,x)=k-1|N(-\delta,0)>0] + O(1)$$

$$\Rightarrow K_X(X)\ \lambda = D_X\ [P_O(x)] \quad \text{as } \delta \to 0 \quad \ldots\ (1.05)$$

where D_X denotes the derivative.

The equation 1.05 links the distribution of the interval between successsive events with survivor function $K_X(x)$ to that of forward recurrence time with survivor function $P_O(x)$ (where $P_O(x)$ is the probability that starting from an arbitrary time instant, there are no events in the following interval of length x.

1.4 PALM'S INTEGRAL EQUATION FOR PALM PROBABILITY AS A LIMITING PROBABILITY :

More general results connecting distributions of events conditional on a point at the origin with those where the origin is an arbitrary instant may be obtained. Assuming that the process is completely stationary and has a finite occurrence parameter ν and is orderly so that ν is equal to the rate λ of the process. Then for each $x > 0$, the palm distribution

is a discrete distribution defined by

$$\pi_K(x) = \lim_{\delta \to 0+} P\,[N(0,x) = k \mid N(-\delta,0) > 0] \quad ..(1.06)$$

for k= 0,1,2,.....

In a careful mathematical development the existence of γ and $\pi_K(x)$, and more generally of other limiting probabilities of events B given $N(-\delta.0)>0$ of the form

$$\lim_{\delta \to 0+} P\,[\,B \mid N(-\delta,0) > 0\,] \quad .....(1.07)$$

has to be proved.

Let the probability measure II be defined for events B on those processes which have a point at the origin. The measure II can then be shown to satisfy

$$II(B) = \lim_{\delta \to 0+} P\,[\,B \mid N(-\delta,0) > 0] \quad .....(1.08)$$

for a wide class of events B. The measure II is called the Palm measure of the process. In equation (1.08) the distribution of the interval measured from an arbitrary time instant to the next point of the process is linked to that of the interval between successive points. In the same way the functions

π_k (x) defining the palm distributions given in (1.06), which specify the distribution of the number of events in an interval of length x given a point at the origin, can be connected with the functions P_k (x) which give the distribution of the number of events in an interval of length x starts at an arbitrary origin. These connecting equations are known as Palm-Khintchine equations and may be derived as follows :

$$\delta^{-1} [p_k(x+\delta) - p_k(x)] = -\delta^{-1} [P[N(-\delta,0)>0]].$$

$$[P(N(0,x))=k| N(-\delta,0)>0)]$$

hence for limit as $\delta \to 0+$ and for right hand derivative D_x

$$D_k [p_k(x)] = \delta^{-1} P[n(-\delta,0)>0] . [\pi_{k-1}(x)]$$

$$= -[\pi_k(x) - \pi_{k-1}(x)] \text{ (from (1.03)\& (1.06) ...(1.09)}$$

The corresponding equation for k = 0 has already been derived and is

$$D_x[p_o(x)] = - \lambda \pi_o(x) \text{ .....(1.10)}$$

It follows from (1.09) and (1.10)

$$-1/ \lambda D_x [p_o(x) + \ldots + p_k(x)] =$$

$$-1/\lambda\, D_x[P[N(x) \leq k]] = \pi_k(x)$$

Therefore, the probability of having exactly k events in (0,x) starting from an event at 0, can be obtained by differentiating the probability of getting not more than k events in (0,x) where 0 is an random time instant. Alternatively from (1.11) and 1.12).

$$p_0(x) + p_1(x) + \ldots\ldots + p_k(x) = P(N(x) \leq k)$$
$$= 1 - \lambda\, P\, \pi_K(u)\, du \quad \ldots\ldots\ldots\ldots (1.13)$$

So that the probability of getting not more than k events in (0,x), where 0 is an random instant can be obtained by integrating the probability of exactly K events in the interval when there is an event at 0 . In addition the right hand side of (1.13) is equal to

$$1 - \lambda \int_0^x \pi_k(u)\, du = \lambda \int_x^\infty \pi_u(u)\, du$$

$$\therefore p_o(x) + p_1(x) + \ldots\ldots + p_k(x) = 1 - \lambda \int_0^x \pi_k(u)\, du$$

$$\Rightarrow p_0(\infty) + p_1(\infty) + \ldots\ldots\ldots + P_k(\infty) = 1 - \lambda \int_0^\infty \pi_k u\, du$$

$$\Rightarrow \lambda \int_0^\infty \pi_k(u)\, du = 1$$

$$1 - \lambda\int_0^x \pi_k(u)\,du = \lambda\int_0^\infty \pi_k(u)\,du - \lambda\int_0^x \pi_k(u)\,du$$

$$= \lambda\int_x^\infty \pi_k(u)\,du$$

Again $\lambda\int_x^\infty \pi_k u\,du = \lambda\int_0^\infty \pi_k\, y+x\,dy$ putting $u=y+x$

Hence $p_o(x) + p_1(x) + \ldots\ldots + p_k(x) =$

$$P[N(x) \le k] = \lambda\int_0^\infty \pi_k\,(y+x)\,dy \quad \ldots\ldots \quad (1.14)$$

(1.14) may be justified by the argument that if '0' is an arbitrary time instant and there are no more than k events in (0,x) then there most exist an event with coordinate y, for some Y > 0, such that there are exactly k events in (-Y, x). Since the process is orderly, the probability of an event in $(-y, -y+\delta)$ is $\lambda\,\delta + O(\delta)$ and therefore

$$P[\,N(x) \le k] = \lambda\int_0^\infty \pi_k(y+x)\,dy$$

The equations (1.09), (1.10) and (1.13) can be summarized by using probability generating functions.

For if it is defined

$$G(Z,x) = \sum_{K=0}^{\infty} Z^k \, p_k \, (x)$$

$$G_0(Z,x) = \sum_{K=0}^{\infty} Z^k \, \pi_k \, (x)$$

so that G refers to an arbitrary origin while G_o refers to the situation given a point at origin.

In fact,

$$\sum_{i=K}^{\infty} \pi_i \, (x) = G^{(k)} \, (x)$$

where the right hand side denotes the cumulative distribution of $[X_1 + X_2 + ... + X_k]$ obtained by k - fold convolution.

That is

$$\pi_k \, (x) = G^{(k)} \, x - G^{(k+1)} \, (x)$$

and the $p_k(x)$ are given by

$$p_k(x) = -\lambda \int_0^x [\pi_k(u) - \pi_{k-1}u] \, du \text{ for } k>0 \quad ....(1.5)$$

$$p_o(x) = 1 - \lambda \int_0^x \pi_o(u) \, du \quad .....(1.6)$$

1.5 BIRTH AND DEATH PROCESS :

One of the obvious generalization of the pure birth processes is to permit X (t) to decrease as well as increase, for example by the death of members. These are more relevant to biological population in which both births and deaths occur. The birth and death process developed here is applied to some problems of policy making in sterilization in chapters 5 of the

thesis.

Let X (t) = Size of the population at time t

with the initial size $X(0) = k_0$

and P (t) = 0 where X (t) = k

Given X (t) = k we assume that the probability of exactly one birth in the interval (t , t+δt) is $\lambda_o(t) + O(\delta)$

(i) The probability of exactly one death is

$$\mu_k(t) + O(\delta)$$

(ii) The probability of more than one change is $O(\delta)$.

Therefore, the probability of no change in (t, t+δt) is

$$1 - \lambda_k(t) - \mu_k(t) + O(\delta).$$

Consequently, the probability $p_k(t+\delta)$ at some time t+δ may be given by the kolmogorov equation

$$P_k(t+\delta) = p_k(t) [\, 1 - (\lambda_k(t) + \mu_k(t) + O(\delta))\,] + p_{k-1}(t) \cdot \lambda_{k-1}(t) \cdot \delta + p_{k+1}(t) \cdot \mu_{k+1}(t) \cdot \delta + O(\delta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.17)$$

This gave us the corresponding system of differential equation

$$dp_o(t) = -[\lambda_o(t)+\mu_o(t)]\, p_o(t) + \mu_1(t)p_1(t) \quad .. \quad (1.18)$$

so on till

$$dp_k(t) = -[\lambda_k(t)+\mu_k(t)]\, p_k(t) + \mu_{k+1}(t)p_{k+1}(t) \quad .. \quad (1.19)$$

for $k \geq 1$

Under initial condition

$$P_{ko}(0) = 1 \text{ and } p_k(0) = 0 \text{ for } k=k_o$$

(1.19) completely determines the probability distribution $P_k(t)$.

Appropriate assumption may be made regarding the function $\lambda_k(t)$ and $\mu_k(t)$ to obtain stochastic process corresponding to empirical phenomena.

A particular case : linear growth

$$\lambda_k(t) = k\lambda \quad , \quad \mu_k(t) = k\mu$$

$$\Rightarrow \quad d\, p_o(t) = \mu p_1(t) \quad \ldots\ldots\ldots\ldots\ldots \quad (1.20)$$

$$\frac{dp_k(t)}{dt} = -\, k\,(k+\mu)p_k(t) + (k-1)\lambda p_{k-1}(t) + (k+1)\mu p_{k+1}(t) \quad \ldots\ldots\ldots\ldots\ldots \quad (1.21)$$

Employing the method of probability generating function (p.g.f.) to solve the same

$$G_X(s,t) = \sum_{k=o}^{\infty} s^k\, p_k(t) \quad \ldots \quad (1.22)$$

From (1.21) it follows

$$\frac{d}{dt} G_X(s,t) + (1-s)(\lambda s - \mu \frac{d\zeta_X}{ds}(s,t) = 0 \quad \ldots \quad (1.23)$$

With the initial condition $t = 0$

$$G_X(s,0) = s^{ko} \quad \ldots\ldots\ldots\ldots\ldots \quad (1.24)$$

The auxiliary equations are

$$dt = \frac{ds}{(1-s)(\lambda s-\mu)} \quad \ldots\ldots \quad (1.25)$$

and $\frac{d}{dt} G_X(s,t) = 0$

For $\lambda = \mu$ we may use partial fractions to rewrite the first auxiliary equation as

$$dt = \frac{\lambda}{(\lambda-\mu)(\lambda s-\mu)}\,ds + \frac{1}{(\lambda-\mu)(1-s)}\,ds \quad \ldots \quad (1.26)$$

$$\Rightarrow \quad (\lambda-\mu)\,dt = \delta \log\left[\frac{\lambda s-\mu}{1-s}\right] \quad \ldots\ldots\ldots \quad (1.27)$$

Integrating both sides of (1.27) =>

$$\left[\frac{1-s}{s-\mu}\right] e^{(\lambda-\mu)t} = \text{constant} \quad \ldots \quad (1.28)$$

The second auxiliary equation of (1.25)

$$G_X(s,t) = \text{constant} \quad \ldots\ldots\ldots\ldots \quad (1.29)$$

Therefore the general solution of (1.23) is

$$G_X(s,t) = \phi\left[\frac{1-s}{\lambda s-\mu}\right] e^{(\lambda-\mu)t} \quad \ldots\ldots \quad (1.30)$$

where ϕ is an arbitrary differentiable function.

Using the initial condition, it can be seen at t = 0

$$\phi\left[\frac{1-s}{\lambda s-\mu}\right] = s^{k} \quad \ldots\ldots\ldots \quad (1.31)$$

at least for all s with $|s| < 1$.

Hence for all θ such that $|1+\theta| < |1+\theta|$

$$\phi(0) = \left[\frac{1+\theta\mu}{1+\theta}\right]^{ko} \quad \ldots\ldots\ldots \quad (1.32)$$

Letting

$$\theta = \frac{1-s}{\lambda s - \mu} e^{(\lambda-\mu)t} \quad \ldots\ldots \quad (1.33)$$

The particular solution for the case of $\mu = \lambda$ given by

$$G_X(s,t) = \left[\frac{(\lambda s-\mu) + \mu(1-s)\, e^{(\lambda-\mu)t}}{(\lambda s-\mu) + \lambda(1-s)\, e^{(\lambda-\mu)t}}\right]^{ko}$$

for $\mu = \lambda$ $\ldots\ldots\ldots$ (1.34)

Putting

$$\alpha(t) = \mu \frac{1 - e^{(\lambda-\mu)t}}{\mu - \lambda\, e^{(\lambda-\mu)t}} \quad \ldots\ldots\ldots \quad (1.35)$$

$$\beta(t) = \frac{\lambda}{\mu}\alpha(t)$$

The p.g.f. can be rewritten as

Expanding the p.g.f. $G_X(s : t)$

$p_k(t) = P[X(t) = k]$ can be written as

$$p_k(t) = \sum_j \begin{bmatrix} k_o \\ j \end{bmatrix} \begin{bmatrix} k_o+k-j-1 \\ k-j \end{bmatrix} + (\alpha(t))^{ko-j} (\beta(t))^{k-j} [1-\alpha(t) -\beta(t)]^j$$

for $k \geq 1$ ..... (1.37a)

and $p_o(t) = [\alpha(t)]^{ko}$ ..... (1.37b)

The use of Martingales based on a density dependent birth and death process is done in chapter 5 on sterilization policy.

A Stochastic Process $\{X_n\} : n = 0, 1, 2, 3, \ldots$ is a martingale if for all $n = 0,1,2,3, \ldots$

i) $E\{|X_n|\} < \infty$

ii) $E\{X_{n+1} \mid X_0, X_1, \ldots, X_n\} = X_n$

..... (1.38)

hold

A more general definition :

Let $\{X_n\} : n = 0,1,2,3, \ldots$ and $(Y : n = 0, 1, 2, 3, \ldots)$ are stochastic processes.

We say $\{X_n\}$ is a Martingale with respect

to $\{ Y_n \}$ if for all n = 0, 1, 2, 3, ...

i) $E \{ | X_n^+ | \} < \infty$

ii) $E \{ X_{n+1} | Y_0, Y_1, \ldots, Y_n \} = Y_n$

..... (1.39)

One may imagine $(Y_0, Y_1, \ldots\ldots Y_n)$ as the information history upto the stage n.

1.6 WALD'S MARTINGALE :

Let $Y_0 = 0$ &

$E [\exp \{ \lambda Y_k \}]$ existing for some $\lambda = 0$ where (Y_k) 's are independent identically distributed random variables for k = 0, 1, 2, 3, ... then

$$X_n = (\phi(\lambda))^{-n} e^{\lambda(Y_1 + \ldots + Y_n)}$$

is a Martingale with respect to Y_n.

1.7 SUB MARTINGALE AND SUPER MARTINGALE:

Let $\{ X_n : n = 0,1,2,\ldots\}$ and $\{ Y_n : n = 0,1,2,\ldots\}$ be stochastic processes then (X_n) is called a Super Martingale with respect to (Y_n) if for all n

i) $E \{ | X_n^- | \} > - \infty$

where $x^- = \inf \{ X, 0 \}$

ii) $E [X_{n+1} | Y_0, Y_1, \ldots, Y_n] \leq X_n$

iii) X_n is a function of $Y_0, Y_1, \ldots, Y_n$.

Similarly (X_n) is called a Sub Martingale with respect

to (Y_n) if :-

i) $E\{X_n^+\} < \infty$ where $X^+ = \text{Sup}(X, 0)$

ii) $E[X_{n+1} \mid Y_0, Y_1, Y_2 \ldots, Y_n] \geq X_n$

iii) X_n is a function of $Y_0, Y_1, \ldots, Y_n$.

1.8 DEFINITION OF MARKOV TIME :

It occurs in many Contexts. A random variable T is called Markov time w.r.t. Yn associated with a Martingale (X_T) for every n with respect to marker chain (Y_n) if T takes the value 0, 1, 2,..... and if for all n = 0, 1, 2,..... the event T = n is determined by $(Y_0, Y_1, \ldots Y_n)$.

By the term 'determined' implied that the indicator function of the event (T=n) cam be written as a function of $Y_0, Y_1, Y_2, \ldots, Y_n$.

1.9 OPTIONAL SAMPLING THEOREM :

Suppose (X_n) is a Martingale and T is a Markov time with respect to (Y_n) then we establish

$$E(X_0) = E(X_{T \cap n}) \quad \ldots (1.40)$$

If $T < \infty$ then $\lim X_{T \cap n} = X_T$;

actually $X_{T \cap n} = X_T$ where $n > T$.

Thus whenever we can justify the interchange of limit n -> ∞ and we can formulate the following :

$$E[X_0] = \lim_{n\to\infty} E[X_{T\cap n}] = E\left[\lim_{n\to\infty} X_{T\cap n}\right] = E(X_T) \quad .. \quad (1.41)$$

1.10 **OPTIONAL STOPPING THEOREM:**

Let (X_n) be a Martingale and T be a Markov time. If

i) $P[\ T < \infty\ ] = 1$

ii) $E[\ |\ X_T\ |\ ] < \infty$

iii) $\lim_{n\to\infty} E[\ X_n\ I_{(T>n)}\ ] = 0$

$$E(X_T) = E(X_0) \quad .... \quad (1.42)$$

where $I_{(\ T\ >\ n)}$ represents the indicator function for T > n.

Counter Theory is employed to evaluate the direct and indirect strategies for the reduction of conception rate.

A Geiger-Muller Counter Model I is a registering mechanism for detecting the presence of a radioactive material arriving at the counter, but because of inertia, the counter will not register some of the impulses. More specifically , suppose that an impulse arrives at a fixed time t the counter registers the impulse. The registration causes a dead time say of

length π_1 and impulses during the dead time will not be registered by the counter. In general, the first impulse to arrive after the termination of the dead time π_1 again will be registered by the counter and this again causes a dead time of length say π_2 and so on. However, in the simpler cases, the impulses which arrive during a dead time do not cause any dead time so that each dead time is caused by a registered impulse. This is called a **Geiger-Muller Counter of type I. (Pyke 1958)**. In a counter of type II (Smith 1958) each arriving impulse causes a dead time so that arrivals during a period of dead time prolong further the dead time. A counter model although basically a description of certain physical processes can be applied to many Biological, Social and Industrial processes.

1.11 REPLACEMENT MODELS:

We illustrate the replacement model dead time using the following problem :

A conception takes place with intensity λ subject to the condition that every conception is followed by infecundable exposure x (fixed dead time) during which no further conception takes place. Then

to obtain the probability distribution of the number of conceptions in (o,t].

Let $\tau_1 < \tau_2 \ldots\ldots \tau_n \ldots\ldots$ be the renewal times (waiting time of conceptions) with conception rate (Poisson intensity) having negative exponential density function.

$$f(t) = \lambda e^{-t\lambda} \quad 0 \leq t < \infty \; ; \lambda > 0$$

$$P[\tau_1 \leq X] = 1 - e^{-\lambda X} \; ; \; \lambda \geq 0 \; ; \; 0 \leq X < \infty$$

$$P[\tau_n - \tau_{n-1} \leq X] = 1 - e^{-\lambda(X-\pi)} \text{ for } X \geq \pi$$

$$= 0 \text{ otherwise} \qquad (1.43)$$

Let W (t,n) = Probability of not more than n events (conceptions) up to time t

$$W(t, n) = P[\tau_{n+1} > t]$$

$$= 1 - P[\tau_{n+1} \leq t]$$

$$= 1 - F_{n+1}(t)$$

$$= R_{n+1}(t) \qquad (1.44)$$

where $F_{n+1}(.)$ is the c.d.f. of the random variable τ_{n+1} and $R_{n+1}(t) = 1-F_{n+1}(t)$ is the corresponding survival function.

Denoting L (.) as the Laplace transform

$$L[W(t, n)] = \int_0^\infty e^{-ts} (1-F_{n+1}(t)\, dt$$

$$= \frac{1}{s} - \frac{1}{s} L [F_{n+1}(t)] \qquad (1.45)$$

$L(F_{n+1}(t)) = \frac{1}{s} L(f_{n+1}(t)]$ where $f_{n+1}(.)$ is the interval density function of $(n+1)^{th}$ renewal (order of conceptions).

We have $f_{n+1}(t) = f_1 * \{f_{(n)}\}*$
where f_1 is the density function on τ_1, f is the density function of $(\tau_r - \tau_{r-1})$, $r \geq 2$, and * stands for convolution; and $f(n)^*$ ≡ n fold convolution of f

$$L [f_{n+1}(t)] = L[\{f_1(t)\} \{L(f(t)\}^n]$$

$$\text{Now } L [f_1(t)] = \frac{\lambda}{\lambda + S}$$

$$\text{and } L [f(t)] = \int_\pi^\infty e^{-st} \lambda\, e^{-\lambda(t-\pi)} dt$$

$$= \frac{\lambda\, e^{-\pi s}}{S + \lambda} \qquad (1.46)$$

$$L [f_{n+1}(t)] = \frac{\lambda^{n+1}}{(S + \lambda)^{n+1}} e^{-n\pi\lambda} \qquad (1.47)$$

Putting (1.46) and (1.47) in (1.45)

$$L\,[W\,(t,\,n)] \;=\; \frac{1}{S} - \frac{1}{S}\,\frac{\lambda^{n+1}}{(S+\lambda)^{n+1}}\,e^{-n\pi s} \qquad (1.48)$$

By taking Inverse Laplace Transform

$$W\,(t,\,n) \;=1 - \lambda\int_{n\pi}^{t} \frac{e^{-\lambda(u-n\pi)}\,\lambda^{n}}{\Gamma(n+1)}\,du \qquad (1.49)$$

Also using the result

$$\sum_{j=0}^{n} e^{-M}\,\frac{M^{j}}{j!} \;=\; 1 - \int_{0}^{M} \frac{e^{-z}\,z^{n}}{\Gamma\,(n+1)}\,dz \qquad (1.50)$$

Putting $M = \lambda\,(t - n\pi)$ on both sides of (1.50)

$$\sum_{j=0}^{n} e^{-\lambda(t-n\pi)}\,\frac{[\,\lambda\,(t-n\pi)\,]^{j}}{j!} \;=1 - \lambda\int_{0}^{(t-n\pi)} \frac{e^{-z}\,z^{n}}{\Gamma\,(n+1)}\,dz$$

Further on substitution of $z = \lambda\,(u-n\pi)$

$z = 0 \quad => u = n\pi$

$z = \lambda\,(t-n\pi) \quad => u = t$

$$\sum_{j=0}^{n} e^{-\lambda(t-n\pi)}\,\frac{[\,\lambda(t-n\pi)\,]^{j}}{j!}$$

$$= \quad 1 - \frac{\lambda}{\Gamma(n+1)}\int_{n\pi}^{t} e^{-\lambda(u-n\pi)}\;[\,\lambda(u-n\pi)^{n}\,]\,du \qquad (1.51)$$

Comparing (1.51) with (1.49)

$$W(t; n) = \sum_{j=0}^{n} e^{-\lambda(t-n\pi)} \frac{[\lambda(t-n\pi)]^j}{j!} \qquad (1.52)$$

$$P[X = n] = W(t; n) - W(t; n-1)$$

$$= \sum_{j=0}^{n} e^{-\lambda(t-n\pi)} \frac{[\lambda(t-n\pi)]^j}{j!}$$

$$- \sum_{j=0}^{n-1} e^{-\lambda[1-(n-1)\pi]} \frac{[\lambda(t-(n-1)\pi)]^j}{j!} \qquad (1.53)$$

and $n \leq [t/\pi]$ where $[t/\pi]$ refers to the greatest integer contained in t/π

1.12 COUNTER MODEL TYPE I (RANDOM VARIABLE DEAD TIME):"

We illustrate the concept of counter model type I with the following problem :

Problem : A particle arrives at t=0 and locks the counter for a dead time of duration Y_1. With the registration of the particle the counter is blocked for a time of length say Y_2. The next particle to be registered is that of the first arrival once the counter is fixed. The process is repeated where the successive locking times denoted by Y_1, Y_2, Y_3 are assumed to be independent with a common distribution

$$P[Y_k < Y] = G(Y)$$

and independent of the arrival process. We obtain the waiting time distribution of the inter-arrival of the process.

Let $Z = Y + Y_Y$.

Where Y is the dead time following the registration and Y_Y is the residual life time before the counter is locked.

Where Y is the dead time and Y_Y be the residual life time before the counter is locked.

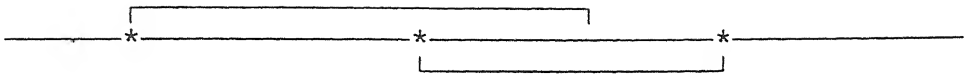
Denoting by $g(.)$ and $f(.)$ the density functions of the dead time and the residual free time (before the system is again being locked) have

$$P[Z \leq Y + Y_Y \leq Z + dz] = \emptyset(z)\, dz$$

$$= \int_0^z g(y)\, f(z-y)\, dy$$

1.13 COUNTER MODEL TYPE II (FIXED DEAD TIME):

Here the locking mechanism is more complicated. As before, an incoming signal is registered if any only if it arrives when the counter is free. In type I counter only recorded particles induced the counter to lock. For Type II Counter every arriving signal can prolong the dead time in the counter, the associated locking times being added concurrently



* ≡ Arrival (registration)

⌜⌝ ≡ Dead time

Figure :

Let $\tau_1 < \tau_2 < \ldots\ldots\ldots \tau_n$ are the registration times of the first, second and the nth renewal and arrivals take place with Poisson rate λ. The length of the dead time following every registration is π.

We assume $[\tau_n - \tau_{n-1}]$ are the independent identically distributed random variables.

$$P[\tau_1 \leq x] = 1 - e^{-\lambda x} \qquad (1.55)$$

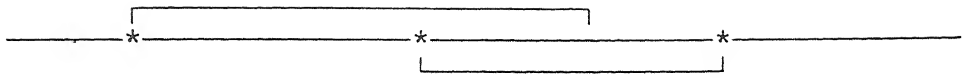
$$P[\tau_n - \tau_{n-1} \leq x] = F(x) \qquad (1.56)$$

where the structure of F(x) is to be obtained.

Let $P[x \leq \tau_n - \tau_{n-1} \leq x + dx] = f(x)\,dx$ (1.57)

$$\phi(s) = \int_0^\infty e^{-st} f(t)dt = L[f(t)]$$

Let $u(t) = dU(t)/dt$ be the renewal density of the process and U (t) be the renewal function.



* ≡ Arrival (registration)

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Figure :

Let $\tau_1 < \tau_2 < \ldots\ldots\ldots \tau_n$ are the registration times of the first, second and the nth renewal and arrivals take place with Poisson rate λ. The length of the dead time following every registration is π.

We assume $[\tau_n - \tau_{n-1}]$ are the independent identically distributed random variables.

$$P[\tau_1 \leq x] = 1 - e^{-\lambda x} \qquad (1.55)$$

$$P[\tau_n - \tau_{n-1} \leq x] = F(x) \qquad (1.56)$$

where the structure of F(x) is to be obtained.

Let $P[x \leq \tau_n - \tau_{n-1} \leq x + dx] = f(x)\,dx$ (1.57)

$$\phi(s) = \int_0^\infty e^{-st} f(t)dt = L[f(t)]$$

Let $u(t) = dU(t)/dt$ be the renewal density of the process and U (t) be the renewal function.

Then the Laplace transform of the Renewal Density is given by

$$L[U(t)] = \frac{L[f_1(t)]}{1-L[f(t)]}$$

$$L[f_1(t)] = \lambda \int_0^\infty e^{-st}\, e^{-\lambda t}\, dt = \frac{\lambda}{\lambda + s} \qquad (1.58)$$

Also $$L[U(t)] = L\frac{dU(t)}{dt} = \frac{\frac{\lambda}{\lambda + s}}{1 - \phi(s)}$$

$$1 - \phi(s) = \frac{\lambda}{\lambda + s}\left[\int_0^\infty [e^{-st}\, \frac{dU(t)}{d(t)}\, d(t)]\right]^{-1}$$

$$1 - \frac{\lambda}{\lambda + s}\left[\int_0^\infty [e^{-st}\, \frac{dU(t)}{d(t)}\, d(t)]\right]^{-1} = \phi(s) \qquad (1.59)$$

To obtain dU(t)/dt we proceed as follows :

$$U(t+dt) - U(t) = e^{-\lambda t}\, \lambda \delta t + (\delta t) \text{ if } t \le \pi \qquad (1.60)$$

given that initially the counter is free.

Also $$U(t+dt) - U(t) = e^{-\lambda \pi}\, \lambda \delta t + 0(\delta t) \text{ if } t \ge \pi \qquad (1.61)$$

$$\frac{dU(t)}{dt} = \begin{cases} \lambda\, e^{-\lambda t} \text{ if } t < \pi & (1.62) \\ \lambda\, e^{-\lambda \pi} \text{ if } t \ge \pi & (1.63) \end{cases}$$

$$\therefore \quad L\left[\frac{dU(t)}{dt}\right] = \int_0^\infty e^{-st}\, \frac{dU(t)}{dt}\, dt$$

$$= \lambda \int_0^{\pi} e^{-st} e^{-\lambda t} dt + \int_{\pi}^{\infty} e^{-st} e^{-\lambda \pi} dt$$

$$= \lambda \int_0^{\pi} e^{-t(\lambda+s)} dt + \lambda e^{-\lambda \pi} \int_{\pi}^{\infty} e^{-st} dt$$

$$L[U(t)] = \frac{\lambda e^{-\lambda(\lambda+s)\pi}}{s} - \frac{\lambda e^{-(\lambda+s)\pi}}{\lambda + s} + \frac{\lambda}{\lambda + s} \qquad (1.63')$$

$$\Rightarrow \quad U(t) = \frac{dU(t)}{dt}$$

$$L^{-1}\left[\frac{\lambda}{\lambda+s} + \lambda e^{-\lambda \pi}\left(\frac{1}{s} - \frac{1}{\lambda+s}\right) e^{-s\pi}\right] \qquad (1.64)$$

$$L^{-1}\left[\frac{\lambda}{\lambda+s}\right] = \lambda e^{-\lambda y}$$

$$L^{-1}\left[\lambda e^{-\lambda \pi}\left(\frac{1}{s} - \frac{1}{\lambda+s}\right)\right] = \lambda e^{-\lambda \pi}(1-e^{-\lambda t})$$

$$L^{-1}\left[\lambda e^{-\lambda \pi}\left(\frac{1}{s} - \frac{1}{\lambda+s}\right) e^{-s\pi}\right] = \lambda e^{-\lambda \pi}(1-e^{-\lambda(t-\pi)})$$

$$\text{if } t > \pi$$

$$= \lambda e^{-\lambda \pi}$$

$$\text{if } t \leq \pi$$

by the second shifting property of Laplace Transform.

$$U(t) = \lambda e^{-\lambda t} + \lambda e^{-\lambda \pi}(1-e^{-\lambda(t-\pi)}) \quad \text{if } t \geq \pi$$

$$= \lambda e^{-\lambda t} + \lambda e^{-\lambda \pi} \quad \text{if } t < \pi \qquad (1.65)$$

$$L[U(t)] = \frac{1}{s} L\left[\frac{dU(t)}{dt}\right] = \frac{1}{s} L[U(t)]$$

$$= \frac{1}{s}\left[-\frac{\lambda}{\lambda + s} + \lambda\frac{1}{s} - \frac{1}{\lambda + s}\right] e^{-(\lambda+s)\pi}$$

$$U(t) = (1-e^{-\lambda t}) + \lambda e^{-\lambda\pi}\left[(t-\pi) - \frac{1}{\lambda}(1-e^{-\lambda(t-\pi)}\right] \quad \text{if } t \geq \pi$$

$$= 1 - e^{-\lambda t} \quad \text{if } t < \pi \qquad (1.66)$$

1.14 COUNTER MODEL TYPE II (RANDOM VARIABLE DEAD TIME):

This counter process is quite difficult to analyse in general form. However, the results are known only for process with Poisson inputs having arrival rate λ.

Let p(t) be the probability that the counter is free at a time t/i and registration is possible.

It is to show that

$$P(t) = e^{-\lambda \int_0^t [1- G(y)]\, dy} \qquad (1.67)$$

where G(y) represents the cumulative density function of the dead time distribution.

Given n occurrences of a Poisson process in the interval (0,t] the distribution of occurrence times is the same as that of n independent random variables taken from a uniform distribution in (0,t].

Proof : The counter is free at time T = t if and only if all dead periods engendered by these signals have been terminated before t.

Let G (t-y) = P (dead time commencing in y will end before the time t).

$$P\left[\begin{array}{l}\text{Induced period}\\ \text{culminated}\\ \text{prior to t}\end{array}\right] = \int_0^t G(t-y)dy \Big/ \int_0^t dy \qquad (1.68)$$

Since the locking times are assumed to be independent and also independent of the arrival process..

We have,

$$P\left[\begin{array}{l}\text{Counter}\\ \text{is free at}\\ \text{time t}\end{array}\ \middle|\ \begin{array}{l}\text{n signals}\\ \text{in (0, t)}\end{array}\right] = \left[\frac{\int_0^t G(t-y)\,dy}{\int_0^t dy}\right] \qquad (1.69)$$

But the number of signals arriving during the interval

(0,t] has a Poisson distribution with mean λt.

From the law of total probability

$$P(t) = \sum_{j=0}^{\infty} \left[\frac{1}{t}\int_0^t G(t-y)dy\right]^j \frac{(\lambda t)^j e^{-\lambda t}}{j!}$$

$$= \sum_{j=0}^{\infty} \left[\frac{\lambda t}{t}\int_0^t G(t-y)dy\right]^j \frac{e^{-\lambda t}}{j!} \qquad (1.70)$$

$$= e^{-\lambda t} \sum_{j=0}^{\infty} \lambda \int_0^t [G(t-y)dy]^j /j!$$

$$= e^{-\lambda t} \; e^{\lambda \int_0^t [G(t-y)dy]}$$

$$= e^{-\lambda\, dy} \; e^{\lambda \int_0^t [G(t-y)dy]}$$

$$= e^{-\lambda\, dy} \; e^{\lambda \int_0^t G(\tau)d\tau}$$

$$P(t) = e^{-\lambda \int_0^t [1-G(y)dy]} \qquad (1.71)$$

Counting with our assumption $\lambda\, p(t)$ is the probability density.

$$\frac{dU(t)}{dt} = \lambda\, p(t)$$

where $U(t)$ represent the renewal function in $(0,t]$ P (a signal appearing in $[t,\ t+\delta t)$].

$$= \quad P(t)\ \lambda\ \delta t + 0\ (\delta t)$$

$$U(t+\delta t) = U(t) + [1.\ \lambda(\delta t)p(t)+0(1-p\ \lambda\delta t)] + 0(\delta t)$$

$$\lim_{t \to 0} \frac{U(t+\delta t) - U(t)}{t} = \lambda P(t)$$

i.e. $$\frac{dU(t)}{dt} = \lambda P(t)$$

$$U(t) = \lambda \int P(\tau)\, d\tau \qquad [\ U(0) = 0)\ ] \qquad (1.72)$$

Putting (1.71) in (1.72)

$$U(t) = \lambda \int_0^t e^{-\lambda [(1-G(y)]dy}\, d\tau \qquad (1.73)$$

This proves the result.

CHAPTER 2

APPLICATION OF PALM PROBABILITY FOR OBTAINING WAITING TIME DISTRIBUTION IN WEIGHTED POISSON PROCESS

2.11 The distribution of $Z_n = \{ T_n - T_{n-1}\}$; $n = 1,2,3...$ where T_n represents the time for the n th renewal in a Poisson process with parameter λ is given by the density function.

$$f(t|\lambda) = \lambda e^{-\lambda t} \; ; \; 0 \le t < \infty; \; \lambda > 0 \qquad (2.1)$$

However, if λ, instead of being a constant assumes a probability distribution with density

$$\psi(\lambda) = \frac{a^k}{\Gamma(k)} e^{-a\lambda} \lambda^{k-1} \qquad (2.2)$$

then the unconditional distribution of time till the first renewal (or arrival) is given by

$$\int_0^\infty f(t|\lambda) \psi(\lambda) \, d\lambda = \frac{k \, a^k}{(a+t)^{k+1}} \qquad (2.3)$$

But the waiting time distribution for the n^{th} arrival given that the first took place at T=t is not certainly the n-fold convolution of (2.3). Because of weighting, the renewal structure of the Poisson process

is completely destroyed leading to interarrival distributions having infinitely divisible structure but with dependent increments. Denoting by x(t) the number of events in (0.t] from a poisson process weighted by a Gamma distribution is given by (2.2) it follows that

$$\text{for } t > s \quad \text{Cov}\,[X(s),\, X(t) - X(s)]$$

$$= E\,[X(s),\, X(t) - X(s) \mid \lambda] - [E(\lambda)]^2\, s(t-s)$$

$$= E\,[\lambda^2\, s(t-s)] - [E(\lambda)]^2\, s(t-s)$$

$$= s(t-s)\ \text{var}\ (\lambda) > 0 \qquad (2.4)$$

Were $\phi(\lambda) = \dfrac{a^k}{\Gamma(k)}\, e^{-a\lambda}\, \lambda^{(k-1)}$; $0 \le \lambda < \infty$, $a,k > 0$

$$E\left(\frac{1}{\lambda}\right)^2 = \int_0^\infty \frac{1}{\lambda^2}\, \phi(\lambda)\, d\lambda$$

$$= \int_0^\infty \frac{1}{\lambda^2}\, \frac{a^k}{\Gamma(k)}\, e^{-a\lambda}\, \lambda^{k-1}\, d\lambda$$

$$= \frac{a^k}{\Gamma(k)} \int_0^\infty e^{-a\lambda}\, \lambda^{(k-2)\,-1}\, d\lambda$$

$$= \frac{a^k}{\Gamma(k)}\ \frac{\Gamma(k-2)}{a^{k-2}}$$

$$= \frac{a^2}{(k-1)(k-2)} \quad ; \ k>2$$

$$f(t_i) = \lambda\, e^{-\lambda t_i} \ ; \ 0 \le t_i < \infty$$

$$E(T_i|\lambda) = \int_0^\infty t_i \lambda e^{-\lambda t_i} dt_i$$

$$= \lambda \int_0^\infty t_i^{2-1} e^{-\lambda t_i} dt_i$$

$$= \frac{\lambda \Gamma(2)}{\lambda^2} = \frac{1}{\lambda}$$

This makes the problem of obtaining the probability distribution of the interarrival time in a compound Poisson process (or any such compound dependent process) somewhat complicated. On the other hand such interarrival time distributions are often considered as very useful for practical purposes; say while obtaining the distribution of interpregnancy (or interbirth) intervals or the waiting time distribution between two morbidity spells or the interarrival time distribution between two accidents. In all the cases the hazard rate $\lambda(t)$ (or the intensity) even for the same t varies between individuals.

Cox and Isham (1980) have dealt with a series of such problems on dependent process classifying the problems according to the nature of dependence. This chapter is devoted to exhibiting an application of Palm

probability to obtain the interarrival distribution between the first and the r^{th} arrival (r = 2.3.....) for a dependent process.

2.1 THE RESULTS:

Defining Palm probability $\psi_r(t)$ as the conditional probability of r events (r = 0, 1, 2, 3.....) in (0,t] given than an event has occurred at time T = O, Khinchine (1960) gives

$$V_r(t) = \frac{k}{a} \int_0^t [\psi_{r-1}(\tau) - \psi_r(\tau)]\, d\tau \qquad (2.5)$$

and

$$V_0(t) = 1 - \frac{k}{a} \int_0^t \psi_0(\tau)\, d\tau \qquad (2.6)$$

where $V_r(t)$ represents the unconditional probability of r events (arrivals) in (O,t] following a weighted Poisson process (weighted by Gamma distribution as in (2.2) and

$$\frac{k}{a} = \lim_{t \to 0} \frac{[1 - V_0(t)]}{t} \qquad (2.7)$$

is the intensity of the process (Khinchine, (1960)]

Then

$$V_0(t) = E_{\lambda}(e^{-\lambda t}/\lambda) = \int_0^{\infty} e^{-\lambda t}\psi(\lambda)d\lambda = \frac{a^k}{(a+t)^k} \quad \ldots\ldots(2.8)$$

Using (2.6) and (2.7) we have

$$\left[\frac{a}{a+t}\right]^k = 1 - \frac{k}{a}\int_0^t \psi_0(\tau)d\tau$$

which on differentiation gives

$$\left[\frac{a}{a+t}\right]^{k+1} = \psi_0(t) \qquad \ldots\ldots (2.9)$$

Again $\psi_0(t) = P[T_1 > t \mid$ an event has occurred at $T = 0]$. Where T_1 is the random time of the first event following the occurrence of an event at $T=0$. Hence

$$f_1(t|.) = \frac{d}{dt}[1-\psi_0(t)] = \frac{(k+1)\,a^{k+1}}{(a+t)^{k+2}} \qquad \ldots (2.10)$$

where $f_1(t|.)$ is the conditional density of the waiting time of the second event given that the first has occurred at $T = 0$. (2.10) is obviously different from the unconditional waiting time density of the first arrival given in (2.3). Proceeding in this way, putting $r = 1$ and differentiating both sides of (2.5)

we have

$$V'_1(t) = \frac{k}{a}\,[\psi_0(t) - \psi_1(t)]$$

$$\frac{d}{dt}\left[K\left(\frac{a}{a+t}\right)^k \left(\frac{t}{a+t}\right)\right] = \frac{k}{a}\left[\frac{a^{k+1}}{(a+t)^{k+1}} - \psi_1(t)\right]$$

$$\Rightarrow \quad \psi_1(t) = \frac{(k+1)\,a^{k+1}\,t}{(a+t)^{k+2}} \qquad \ldots.(2.15)$$

Then,

$$F_2(t|.) = 1-\psi_0(t) - \psi_1(t) \qquad (2.12)$$

where $F_2(t|.)$ is the conditional cumulative distribution function (c.d.f.) of the waiting time distribution of the second arrival given than at T=0 the first arrival took place. Then,

$$F_2(t|.) = \frac{a^{k+1}\,(k+1)}{(a+t)^{k+2}} - \frac{a^{k+2}\,(k+2)}{(a+t)^{k+3}} - \frac{ka^{k+1}}{(a+t)^{k+2}}$$

$$+ \frac{k(k+2)t\,a^{k+1}}{(a+t)^{k+2}} + \frac{a^{k+1}\,(k+1)}{(a+t)^{k+2}}$$

$$= \frac{ta^{k+1}\,(k+1)\,(k+2)}{(a+t)^{k+3}} \qquad (2.13)$$

which is the interarrival density function between the first and the third arrival given that the first

arrival took place at T=0 in a compound Poisson process. Proceeding precisely in the same way we have.

$$V'_2(t) = \frac{k}{a}\left[\dot{\psi}_1(t) - \dot{\psi}_2(t)\right] \qquad (2.14)$$

$$\frac{d}{dt}\left[\frac{(k+1)}{2}\left(\frac{a}{a+t}\right)^k\left(\frac{t}{a+t}\right)^2\right] = \frac{k}{a}\left[\frac{(K+1)a^{k+1}}{(a+t)^{k+1}} - \ddot{U}_2(t)\right].$$

$$\dot{\psi}_2(t) = \frac{t^2\, a^{k+1}\, (k+1)\, k+2}{2!\ (a+t)^{k+3}} \qquad (2.11)$$

$$F_3(t|.) = 1 - \left[\dot{\psi}_2(t) + \dot{\psi}_1(t) + \dot{\psi}_0(t)\right]$$

$$1 - \left[\frac{a^{(k+1)}(K+1)(k+2)t^2}{2!(a+t)^{k+3}} + \frac{a^{k+1}\, t(t+k)}{(a+t)^{k+2}} + \frac{a^{k+1}}{(a+t)^{k+1}}\right]$$

when we get

$$f_2(t|.) = \frac{a^{k+1}\ (k+2)\ (k+3)t^2}{2!(a+t)^{k+4}} \qquad (2.16)$$

using

$$f_n(t|.) = \frac{d}{dt}\left[1 - \sum_{r=0}^{n-1} \dot{\psi}_r(t)\right] \qquad (2.17)$$

and the recurrence relation

$$f_n(t|.) = f_{n-1}\ (t|.) - \dot{\psi}_{n-2}(t) + \frac{a}{k} V_{n-1}(t)\ ] \qquad (2.18)$$

we have

$$f_n(t|.) = \frac{t^{n-1}\, a^{k+1}\, (k+1)\, (k+2)}{(n-1)\, !\, (a+t)^{k+n+1}} \qquad (2.19)$$

which provides the distribution of the time between the first and the $(n+1)^{th}$ arrival given that the first arrival occurred at T=0 for a compound Poisson process weighted by a Gamma distribution.

The treatment for other dependent processes is precisely the same as illustrated in the foregoing example.

REMARKS:

If $(T_i - T_{i-1})$ is the waiting time for the i^{th} order of conception (i = 2,3,.....) or $(T_i - T_{j-1})$ may be called 'Interconception' interval and the conception rate λ (even taken independent of i) varies from individual to individual conforming to some probability distribution say Gamma distribution, then because of waiting λ.

(i) Renewal intervals $(T_i - T_{i-1})$ i = 2,3,..... will cease to become i.i.d.r.v.'s (or i.d.r.v.'s) contrary to the traditional assumption.

(ii) The num. ber of renewals even in two non overlapping intervals will be correlated. The extent of correlation because of the weighting of the process. The renewal structure is completely destroyed; leading to the process to conform to infinitely divisible distribution with dependent increments.

CORRELATION BETWEEN T_i AND $T_{j-i} = T_j - T_i$ $(j > i)$:

Let us assume the probability distribution of λ to be

$$\phi(\lambda) = \frac{a^k}{\Gamma(k)} e^{-a\lambda} \lambda^{k-1}; \quad 0 \leq \lambda < \infty, \quad a, k > 0$$

We have

$$\text{Cov}(T_i, T_j) = E[E(T_i T_j | \lambda)] - E[\{E(T_i | \lambda)\}\{E(E(T_j | \lambda)\}]$$

$$E(T_i T_j | \lambda) = E[T_i (T_j + T_{j-i} | \lambda)]$$

$$= E(T_i^2 | \lambda) + [\text{Cov}(T_i, T_{j-1} | \lambda) + E(T_i | \lambda) E(T_{j-i} | \lambda)]$$

Now $\text{Cov}(T_i, T_{j-i}) = 0$

Since T_i and T_{j-i} being two non overlapping intervals.

$$E(T_i^2 | \lambda) = \frac{i(i+1)}{2}, \quad E(T_i | \lambda) = \frac{1}{\lambda}$$

and $$E(T_{j-i} | \lambda) = \frac{j-i}{\lambda}$$

$$E(T_i T_j | \lambda) = i(i+1) \; E\left(\frac{1}{\lambda^2}\right)$$

since $E\left(\frac{1}{\lambda^2}\right) = \frac{a^2}{(k-1)(k-2)}$; $k>2$

$$E(T_iT_j) = \frac{i(j+1)\,a^2}{(k-1)(k-2)}$$

$$\text{Cov}(T_iT_j) = \frac{a^2\, i(j+k-1)}{(k-1)^2(k-2)};\ k>2$$

$$\text{Var}(T_i) = \frac{a^2\, i(i+k-1)}{(k-1)^2(k-2)};\ k>2$$

$$\text{Cov}(T_i,T_j) = \text{Cov}(T_2,\ T_i + T_{j-i})$$

Next we derive $\text{Cov}(T_i,T_{j-1}) = \text{Cov}(T_i,\ T_i+T_{j-1})$

$$= E(T_i,(T_i+T_{j-1})) - E(T_i)\ E(T_i+T_{j-1})$$

$$= E(T_i^2)+E(T_iT_{j-1})-[E(T_i)]^2- E(T_i\ E(T_{j-1})$$

$$= \text{Var}(T_i) +\text{Cov}(T_i,T_{j-1})$$

$$\text{Cov}(T_i,T_{j-1}) = \text{Cov}(T_i,T_j) - \text{Var}(T_i)$$

$$= \frac{a^2\, i(j-1)}{(k-1)^2(k-2)},\quad k > 2$$

Also $\text{Var}(T_{j-1}) = \text{Var}(T_j) + \text{Var}(T_i) - 2\text{Cov}(T_i,T_j)$

$$= \frac{a^2\ (j-1)\ (j-i+k-1)}{(k-1)^2\ (k-2)}$$

Thus $Cor(T_i, T_{j-1}) = \dfrac{i(j-i)}{\sqrt{(i(i+k-1))}\ \sqrt{(j-1)(j-i+k-1)}}$

which is independent of a.

Now assuming λ to follow ß distn.

We have $\phi(t) = \dfrac{1}{ß(a,k)} \dfrac{\lambda^{a-1}}{(1+\lambda)^{a+k}}$; $0 \leq \lambda \leq \infty$ and $k>0$.

Thus $Cov(T_i, T_j) = E[E(T_i T_j|\lambda)] - E[E(T_i|\lambda)]\ E[E(T_j|\lambda)]$

Now we have to estimate

$E(T_i T_j|\lambda)] = E[T_i(T_j+T_{j-1}|\lambda)]$

$= E(T^2{}_i|\lambda) + [Cov(T^2{}_i, T_{j-1}|\lambda) + E(T_i|\lambda)\ E(T_{j-1}|\lambda)]$

Where $Cov(T_i, T_{j-1}) = 0$

As T_i and T_{j-1} being two non overlapping intervals and

$E(T^2{}_i) = \dfrac{i(i+1)}{\lambda^2}$

$E(T_i) = \dfrac{i}{\lambda}$ and $E(T_{j-1}|\lambda) = \dfrac{j-1}{\lambda}$

$E(T_i, T_j|\lambda) = i(i+1)\ E\left(\dfrac{1}{\lambda^2}\right)$

Where $E\left(\dfrac{1}{\lambda^2}\right) = \dfrac{1}{ß(a,k)} \int_0^\infty \dfrac{1}{\lambda^2} \dfrac{\lambda^{a-1}}{(a+k)^{a+k}}\ d\lambda$

$= \dfrac{1}{ß(a,k)} \int_0^\infty \dfrac{1}{\lambda^2} \dfrac{\lambda^{a-2} - 1}{(a+k)^{(a-2)+(k+2)}}\ d\lambda$

$$= \frac{1}{\beta(a,k)} \beta(a-2,k+2)$$

$$= \frac{\Gamma(a-2)\ \Gamma(k+2)}{\Gamma(a)\ \Gamma(k)}$$

$$= \frac{k\ (k+1)}{(a-1)(a-2)} \quad ; \text{ for all } a>2$$

Thus $E(T_i T_j|\lambda) = \dfrac{i(i+1)k(k+1)}{(a-1)\ (a-2)}$

Thus $Cov(T_i, t_j) = \dfrac{i(i+1)\ k(k+1)}{(a-1)(a-2)} - E\left[\frac{i}{\lambda}\right] E\left[\frac{j}{\lambda}\right]$

therefore $Cov(T_i, T_j) = \dfrac{i(i+1)k\ (k+1)}{(a-1)\ (a-2)} - ij\ [E(\frac{1}{\lambda^2})^2]$

where $E(\frac{1}{\lambda}) = \int_0^\infty \frac{1}{\lambda}\ \phi(\lambda)\ d\lambda$

$$= \frac{1}{\beta(a,k)} \int_0^\infty \frac{1}{\lambda} \frac{\lambda^{a-1}}{(1+\lambda)^{a+k}} d\lambda$$

$$= \frac{1}{\beta(a,k)} \int_0^\infty \frac{\lambda^{(a-1)-1}}{(1+\lambda)^{a+k}} d\lambda$$

$$= \frac{1}{\beta(a,k)} \beta(a-1,\ k+1)$$

$$= \frac{1\ \Gamma(a-1)\ k\ \Gamma(k)}{(a-1)\ \Gamma(a-1)\ \Gamma(k)}$$

$$= \frac{k}{(a-1)} \quad ; \ a > 1$$

$$\text{Cov}(T_i, T_j) = \frac{i(i+1)\ k(k+1)}{(a-1)\ (a-2)} - ij\ \frac{k^2}{(a-1)^2}$$

$$\text{Cov}(T_i, T_{j-1}) = \text{Cov}(T_i, T_j) - \text{Var}(T_i)$$

$$= \frac{i(i+1)\ k(k+1)}{(a-1)\ (a-2)} - \frac{ij\ k^2}{(a-1)^2} - \text{Var}(T_i)$$

Where $V(T_i) = E[E(T_i^2|\lambda)] - E[E(T_i|\lambda)]$

$$= E\left(\frac{i(i+1)}{\lambda^2}\right) - E\left(\frac{i^2}{\lambda^2}\right)$$

$$= \frac{i^2+i}{\lambda^2} - i^2\ E\left(\frac{1}{\lambda^2}\right)$$

$$\Rightarrow V(T_i) = E\left(\frac{i^2 + i}{\lambda^2}\right) - \frac{i^2\ k(k+1)}{(a-1)(a-2)}$$

$$= \frac{i(i+1)\ k(k+1)}{(a-1)(a-2)} - \frac{i^2\ k(k+1)}{(a-1)(a-2)}$$

$$= \frac{ik(k+1)}{(a-1)(a-2)}\ [\ (i+1) - i]$$

$$= \frac{ik(k+1)}{(a-1)\ (a-2)}$$

Therefore $\text{Cov}(T_i, T_{j-1}) = \text{Cov}(T_i, T_j) - \text{Var}(T_i)$

$$= \frac{i(i+1)k(k+1)}{(a-1)\ (a-2)} - \frac{ij\ k^2}{(a-1)^2} - \frac{ik(k+1)}{(a-1)(a-2)}$$

$$= \frac{i^2\ k(k+1)}{(a-1)\ (a-2)} - \frac{ij\ k^2}{(a-1)^2}$$

$$\therefore \text{Cor}(T_i, T_{j-1}) = \frac{\text{Cov}(T_i,\ T_{j-1})}{\sqrt{\text{Var}(T_i)}\ \sqrt{\text{Var}(T_{j-1})}}$$

Where $\text{Var}(t_{j-1}) = E[E(T_{j-1}{}^2|\lambda)] - E[E(T_{j-1}|\lambda)^2]$

$$= E\left[\frac{(j-1)j}{\lambda^2}\right] - E\left[\frac{(j-1)^2}{\lambda^2}\right]$$

$$= j(j-1)\ E\left[\frac{1}{\lambda^2}\right] - (j-1)^2\ E\left[\frac{(j-1)^2}{\lambda^2}\right]$$

$$= [j(j-1) - (j-1)^2]\ E\left[\frac{1}{\lambda^2}\right]$$

$$= [j-1]\ E\left[\frac{1}{\lambda^2}\right]$$

$$\text{Var}(T_{j-1}) = \frac{(j-1)\ k(k+1)}{(a-1)(a-2)}$$

$$\text{Cor}(T_i, T_{j-1}) = \frac{i^2k(k+1)(a-1) - (a-2)\ ijk^2}{(a-1)\ \sqrt{\ k(k+1)i(j-1)}}$$

In this case Conception rate λ follows ß distribution of second kind with parameters a and k then its expectation is given by

$$E(\lambda) = \frac{a}{k-1} \quad ; \; k > 1$$

that means average conception depends on the parameters a and k. Hence $E(\lambda)$ is directly proportional to a whereas indirectly proportional to k.

Our results shows that correlation between two non-overlapping intervals is higher when λ follows ß Beta distribution as compared to the case when λ follows Gamma distributions. In particular we can compare the correlations between 1^{st} and 2^{nd} order of conception. i.e for i=1 , j=2 when λ follows gamma

distribution $\quad Corr(T_1, T_2 - T_1) = \frac{1}{k}$

and in case λ follows ß distribution,

distribution $\quad Corr(T_1, T_2 - T_1) = \frac{[(k+1)(a-1) - 2(a-2)k]k}{(a-1)\sqrt{k(k+1)}}$

CHAPTER - 3

COMPARISON OF COHORT FERTILITIES USING PALM PROBABILISTIC TECHNIQUE

3.0 In this chapter a suitable methodology is evolved to predict and then to compare the future fertility performance. Given the time of occurrence of first birth as t_1 and t_2 respectively of the two cohorts, whose age of marriage differs by a time lag t, it is proposed to analyse the level of fecundability changes between two cohorts by the average number of births to these cohorts of women given that the first birth to the first cohort of women took place at t_1 and the first birth of the second cohort of women took place at $t_2 (t_2 > t_1)$ leaving the residual fertility span $(T-t_1)$ and $(T-t_2)$ respectively for the two classes of women who complete their fertility span at the age T.

The purpose is to see whether postponement of the age at marriage is effective in reducing the fertility status by using the palm probability technique.

3.1 PALM PROBABILITY:

Stochastic modelling involving prediction of the number of events during a fixed period of time (0,t) or the prediction of the waiting time distribution between

two consecutive events or between any two events say the ith and the jth event can be solved by an entirely new technique known as "Palm Probability". It is the conditional probability of a specific number of events given that the event has occurred at the beginning of the interval.

Let us consider two consecutive intervals of length s and (t-s) respectively as shown in the following figure :

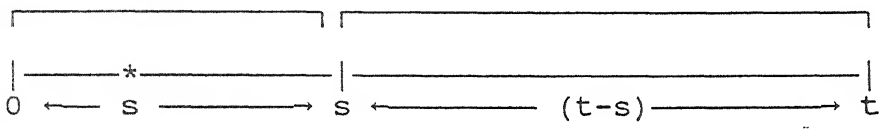

Figure 3.2

Further let the interval (0,s) be so small such that at most one arrival of the event with compound Poisson inputs (i.e. Poisson input λ weighted by a two parameter family of Gamma distribution for taking the variation of fecundability between individuals) is possible and let us assume that ther e is only one arrival in (0,s) without any loss of generality. Then as $s \to 0$, the point of arrival denoted by X (in Figure 3.2) in the limiting position will be located in the beginning of the interval (s,t). What is the probability of having a finite number of arrivals of

events, say k number of arrivals (k=0,1,2,....) in the interval (s,t) of any finite length given an arrival at the beginning of s. What is the waiting time distribution of next arrival or the nth arrival in sequence following that event (i.e. an event has occurred at s)?. These limiting conditional probabilities are known as **Palm Probabilities.**

Let $\phi_k(t)$ = The conditional probability of k number of births in [0,t] given that first birth has occurred at T=0 [$\phi_k(t)$ is a plam probability measure V k=0,1,2,...n] and

$V_k(t)$ = The unconditional probability of k births in (0,t); k=0,1,2,.....n and let the births occur with Poisson intensity λ where λ varies from individual to individual following Gamma distribution.

Then Palm's integral equations connecting $V_k(t)$ and $\phi_k(t)$ are as follows (Khintchine (1960)

$$V_k(t) = \int_0^t [\phi_{k-1}(t) - \phi_k(t)]dt \qquad k = 0, 1, 2,.....n \qquad (3.1)$$

and

$$V_o(t) = h(t) = h(t) \int_0^t \phi_o(t)dt \qquad (3.2)$$

where $h(t)$ represents the intensity of the hazard rate of having a birth at the age t.

Since the hazard rate $h(t)$ is a increasing function of t for a given woman with fecundability level λ we write

$$h(t|\lambda) = \lambda e^{\delta t} \quad \text{where } s > 0$$

as the conditional hazard rate.

It follows that $s = \frac{d \log h(t)}{dt}$

Where s is the time elasticity of the hazard rate i.e the measure of the percentage increases in the hazard rate with 1% measure in time or age and $h(t)/t = o = \lambda$.

For given λ, we have time dependent poisson process $V_k(t|\lambda) = \frac{[\lambda e^{st}]^k}{k!} dt \quad [e^{-\lambda e^{\int_0^t st}} dt]$

which is obtained by solving the differential equation

$$V_k(t+\Delta(t)) = V_{k-1}(t) [\lambda e^{st} + O\Delta(t)] + V_k(t) [1-\lambda e^{st} + O\Delta(t)]$$

$V_k(t)$ being the probability of occurance of k conceptions or events in time $(0,t]$.

$V_o(t|\lambda)$ = Prob. [no birth upto time t]

$$= e^{-\int_0^t h(t/\lambda)\,dt}$$

$$= e^{-\int_0^t \lambda\, e^{\delta t}\, dt}$$

$$= e^{-\lambda \mid e^{\delta t}\mid_0^t}$$

$$= e^{-\lambda\,(e^{\delta t}-1)}$$

$$= e^{-\lambda\, A(t)}$$

where $A(t) = \frac{1}{\delta}\,(e^{\delta t-1})$

$$V_o(t) = \int V_o(t|\lambda)\,\phi(\lambda)\,d\lambda$$

$$= \int_0^\infty e^{-\lambda A(t)}\,\frac{a^k\, e^{-a\lambda}\, \lambda^{k-1}}{\Gamma(k)}$$

$$= \frac{a^k}{\Gamma(k)} \cdot \int_0^\infty e^{-a+A(t)\lambda}\, \lambda^{k-1}\, d\lambda$$

$$= \frac{a^k}{(a+A(t))^k}$$

where $A(t) = \frac{1}{\delta}\,(e^{\delta t} - 1)$

$$\lim_{t\to 0} 1 - V_0(t) = \lim_{t\to 0} \frac{1 - \left[\frac{a}{a + A(t)}\right]^k}{t}$$

$$= \frac{(-k)\ (a)^k}{a^{k+1}}$$

$$= \frac{-k}{a}$$

Since $\lim_{t\to 0} A(t) = \lim_{t\to 0} [\frac{1}{\delta} [\ 1-e^{\delta t}]]$

$= 0.$

$\lim_{t\to 0} A'(t) = -\ e^{\delta t} = -1.$

Which is independent of t showing the stationarity of the process.

Also

$$\frac{d\ V_0(t)}{dt} = d[\ \frac{a}{a + A(t)}\ ]^k$$

$$= \frac{-k\ a^k\ A'(t)}{(a+A(t))^{k+1}}$$

By differentiating Palm's integral equation (3.2)

$$\frac{d}{dt} V_0(t) = - \frac{k}{a} \phi_0(t)$$

$$\Rightarrow \phi_0(t) = - \frac{a}{k} - \frac{dV_0(t)}{dt}$$

$$\Rightarrow \phi_0(t) = \frac{-\ a^{k+1}\ e^{\delta t}}{(a+1\ (e^{\delta t}-1))^{k+1}}$$

$$= \frac{a^{k+1}\ A'(t)}{(a+A(t))^{k+1}}$$

where $A'(t) = -e^{\delta t}$ ....(3.8')

Since $(V_n(t)|\lambda)$ conforms to a differential difference equation given by $n = 0,1,2,\ldots,n$.

We have

$$V_n(t+\delta t|\ \lambda) = V_{n-1}(t)\ [h(t|\ \lambda)\ e^{\delta t} + o(\delta t)] + V_n(t)\ [1-[h(t|\ \lambda)\ e^{\delta t} + o(\delta t)]] \quad (3.10)$$

$$\frac{d}{dt}V_n(t|\ \lambda) = \lambda\ e^{\delta t}\ [V_{n-1}(t|\ \lambda) - V_n(t|\ \lambda)]$$

We have

$$V_n(t|\ \lambda) = \frac{[\ \lambda \int_0^t e^{\ \delta t}\ dt]^n}{n\ !}\ e^{-\lambda \int_0^t e^{\ \delta t}\ dt} \quad (3.11)$$

Again by Palm's integral equation.

for $k = 1$.

$$\frac{d}{dt}V_1(t|\ \lambda) = \lambda\ e^{\delta t}\ [V_o(t|\ \lambda) - V_1(t|\ \lambda)]$$

$$V_1(t|\ \lambda) = \frac{k}{a}\ A(t)\ \left[\frac{a}{a + A(t)}\right]^{k+1} \quad (3.12)$$

Where $A(t)$ is given in (3.6)

$$\phi_1(t) = \phi_o(t) - \frac{a}{k}\frac{d}{dt}V_1(t)$$

$$= -\ \frac{(k+1)\ a^{k+1}\ A(t)A'(t)}{1!\ [a + A(t)]^{k+2}} \quad (3.13)$$

Precisely in a similar way

$$V_2(t) = - \frac{k\ (k+1)\ A(t)^{\ 2}\ a^k}{2!\ [a + A(t)]^{k+2}}$$

$$\phi_2(t) = - \frac{(k+1)\ (k+2)\ a^{k+1}\ A'(t)\ [A(t)]^2}{2!\ [a + A(t)]^{k+3}} \quad (3.14)$$

$$V_3(t) = - \frac{k(k+1)\ (k+2)\ a^k\ [A(t)]^3}{3!\ [a + A(t)]^{k+3}} \quad (3.15)$$

$$\phi_3(t) = - \frac{(k+1)\ (k+2)\ (k+3)\ a^{k+1}\ A'(t)\ [A(t)]^3}{3!\ [a + A(t)]^{k+4}} \quad (3.16)$$

Proceeding in this way we get

$$\phi_n(t) = - \frac{(k+1)\ (k+2)....(k+n)\ a^{k+1}\ A'(t)\ [A(t)]^n}{[a + A(t)]^n} \quad (3.17)$$

Expected number of births in the residual fertility span $(T-t_1)$ given that the first birth has occurred at $t=t_1$ for the first cohort is

$$\sum_{n=0}^{\phi} n\ \phi_n^{(1)}\ (T-t_1)$$

$$= \left[1 - \frac{A(T-t_1)}{a_1+A(T-t_1)}\right]^{-(K+2)}_{1} \frac{a_1^{K+1}\ A(T-t_1)(K_1+1)\ A'(T-t_1)}{(a_1 + A(T-t_1)}$$

$$= f_1\ [a_1,\ k_1,\ \delta,\ (T-t_1)] \quad (3.18)$$

and the expected number of births in the residual fertility span, given that the first birth has occurred at $t=t_2$ for the second cohort is

$$\sum_{n=0}^{\phi} n\,\phi_n^{(2)}\,(T-t_2)$$

$$= \left[1 - \frac{A(T-t_2)}{a_2+A(T-t_2)}\right]^{-(K_2+2)} \frac{a_2^{K+1}\,A(T-t_2)(K_2+1)\,A'(T-t_2)}{(a_2 + A(T-t_2)}$$

$$= f_2\,[a_2,\ k_2,\ \delta,\ (T-t_2)] \qquad (3.19)$$

where $(a_i,\ k_i)$, $i = 1, 2$ are parameters of the Gamma distribution corresponding to the cohorts I and II respectively.

Therefore for a change of age at marriage by

$= t_2-t_1$ for $t_2>t_1$

there is change in the average level of births

$$E = \left[f_1(a_1,\ k_1,\ T-t_1,\ \delta - f_2(a_2,\ k_2,\ T-t_2,\ \delta \right] \qquad (3.20)$$

$f_1 > f_2$ indicates fall in the overall fertility level

$f_1 < f_2$ indicates increase in the fertility level; and

$f_1 \approx f_2$ indicates the stationarity in the fertility level.

3.2 ESTIMATION OF THE PARAMETERS OF THE MODEL :

We have for the first cohort $E_1[X(T-t_1)|$ there is

a birth at $t=t_1$]

$$= \left[1 - \frac{A(T-t_1)}{a_1+A(T-t_1)}\right]^{-(K_1+2)} a_1^{K+1} \frac{A(T-t_1)(K_1+1)\ A'(T-t_1)}{(a_1 + A(T-t_1)}$$

$$= f_1\ [a_1,\ k_1,\ \delta,\ (T-t_1)] \quad \text{say} \qquad (3.18)$$

and for the second cohort

$E_2[X(T-t_2)|$ there is a birth at $t=t_2]$

$$= \left[1 - \frac{A(T-t_2)}{a_2+A(T-t_2)}\right]^{-(K_2+2)} a_2^{K+1} \frac{A(T-t_2)(K_2+1)\ A'(T-t_2)}{(a_2 + A(T-t_2)}$$

$$= f_2\ [a_2,\ k_2,\ \delta,\ (T-t_2)] \quad \text{say} \qquad (3.19)$$

Then the variance of the number of births for the first cohort

$Var[X(T-t_1)|$ birth at $t=t_1$ =

$$\sum_{n=0}^{\phi} n^2\ \phi_1^{(1)}\ (T-t_1) - \left[\sum_{n=0}^{\phi} n\ \phi_1^{(2)}\ (T-t_1)\right]^2$$

$$= \frac{(k_1+1)a_1^{K+1}\ A'(T-t_1)\ A(T-t_1)}{a_1+A(T-t_1)} \left[\frac{a_1}{(a_1+A(T-t_1)}\right]^{-(k_2+2)}$$

$$+ \frac{a_1^{K_1+1}\ A'(T-t_1)(K_1+1)(k_1+2)}{a_1+A(T-t_1)} \left[\frac{A(T-t_1)}{(a_1+A(T-t_1)}\right]^2$$

$$\left[\frac{a_1}{(a_1+A(T-t_1)}\right]^{-(k_2+3)}$$

$$-\left[\frac{a_1^{K_1+1}\, A(T-t_1)(k_1+1)A'(T-t_1)}{a_1+A(T-t_1)} \left|\frac{a_1}{(a_1+A(T-t_1)}\right|^{-(k_1+2)}\right]^2$$

$$= f_3\,[a_1,\ k_1,\ \delta,\ (T-t_2)] \qquad \text{say} \quad (3.21)$$

and the variance of the number of births corresponding to the second cohort

$$\sum_{n=0}^{\phi} n^2\, \phi_1(2)\ (T-t_2) - \left[\sum_{n=0}^{\phi} n\, \phi_1(2)\ (T-t_2)\right]^2$$

$$= \frac{(k_2+1)a_2^{K_2+1}\, A'(T-t_2)}{a_2+A(T-t_2)}\left[\frac{a_2}{(a_2+A(T-t_2)}\right]^{-(k+1)}$$

$$+ a_2^{K_2+1}\, A'(T-t_2)(K_2+1)(k_2+2)\left[\frac{A(T-t_2)}{(a_1+A(T-t_2)}\right]^2$$

$$\left[\frac{a_2}{(a_2+A(T-t_2)}\right]^{-(k_2+3)}$$

$$-\left[\frac{a_2K_2+1\ A(T-t_2)(k_2+1)A'(T-t_2)}{a_2+A(T-t_2)}\left[\frac{a_2}{(a_2+A(T-t_2)}\right]^{-(k_2+2)}\right]^2$$

$$= f_4\,[a_2,\ k_2,\ \delta,\ (T-t_2)] \qquad \text{say} \quad (3.22)$$

Now by the method of moments (3.18) and (3.21) for given δ and $T-t_1$ will enable us to estimate a_1 and k_1

by successive approximations..

Thus by the same procedure (3.19) and (3.22) for given δ and $T-t_2$ the estimates of a_2 and k_2 will be obtained.

On substitution of the estimates a_1, k_1, a_2, k_2 and δ for given $(T-t_1)$ and $(T-t_2)$ one can examine the increasing or decreasing trend of fertility as observed in two cohorts who differ by t years at age at marriage. The inherent difference in the fertility, apart from the availability of total marital exposure, if any, is expressed in the estimated values of the parameters (a_i, k_i) $i = 1, 2$ for the two cohorts.

3.3 A NUMERICAL ILLUSTRATION:

When $t_1 = 16$, $t_2 = 18$, $T_1 = 45$

Thus

$E_1[X(T-t_1)|t_1] = f_1(a_1,k_1,\delta,T-t_1)$ = (A)=3.64,4.48,5.50

$Var_1[X(T-t_1)|t_1]=f_3(a_1,k_1,\delta,T-t_1)$=(B)=10.49,9.65,10.69

$T-t_1 = 45-16 = 29$ years

$E_2[X(T-t_2)|t_2] = f_2(a_2,k_2,\delta,T-t_2)$ = (C)=3.09,4.40,4.62

and

$Var_2[X(T-t_2)|t_2]=f_4(a_2,k_2,\delta,T-t_2)$=(D)=9.43,15.20,7.90

$T-t_2 = 45-18 = 27$ years.

Solving (A) and (B) for the given set of hypothetical values as given above by starting with an approximate initial solution of $a_1 = a_1^0$ and $k_1 = k_1^0$ we get by successive iteration

$a_1 = 0.35,\ 0.40$ $\quad k_1 = 1.0,\ 1.5,\ 2.0$

Similarly starting with an approximate initial solution of $a_2 = a_2^0$ and $k_2 = k_2^0$ we solve (C) and (D) by successive iterations and get

$a_2 = 0.35,\ 0.35$ $\quad k_2 = 1.0,\ 2.0,\ 2.0$

Substituting the estimates of a_1, k_1 and a_2, k_2 in (3.20) we get three values of

$E = 0.27,\ -0.24,\ 0.48$ respectively

which shows that in the hypothetical exercise the first cohort has the fertility status not significantly different than that of the second one.

3.4 TREND OF THE PALM PROBABILITY DISTRIBUTION (NUMBER OF BIRTHS DURING FIXED MARITAL EXPOSURE)

A comparison of the mean and the variance of the Palm Probability distribution given in (3.17) shows that

$$\text{Variance} = \text{Mean} + c_i\ (\text{Mean}) - (\text{mean})^2$$

where

$$c_i = (k+3)\frac{(1 - e^{\delta(T-t_i)})}{[a + (1 - e^{\delta(T-t_i)})]}\left[\frac{a}{[a + e^{\delta(T-t_i)}]}\right]^{-1}$$

$$= \left[\left(\frac{k+2}{a}\right)\frac{1}{\delta}(1- e^{\delta(1-e-\delta(T-t_i))})\right], \quad i = 1, 2$$

If k, δ and a satisfy a relation so that $c_i = \Theta$ (mean).

For the non-negativity of the variance it is necessary that $\Theta \geq 1$ for Mean > 1.

= Variance = Mean [1 + c_i - (mean)]

Variance = Mean [1 + {mean (Θ-1}]

In particular, it may be noted that if mean is greater than one, if $0 \leq \Theta < 1$, Variance < Mean; or $\Theta = 1$

=> Variance = Mean.

$0 < \Theta$ = Variance > Mean. For $\Theta = 2$, Variance = Mean = $(\text{Mean})^2$.

Thus in this flexible model by suitable choices of the parameters, the mean/variance ratio can be made in the same order as may be observed in the sample taken.

CHAPTER - 4

THE WAITING TIME DISTRIBUTION BETWEEN THE FIRST AND SECOND ORDER CONCEPTIONS USING PALM PROBABILITY TECHNIQUE

4.1 Biswas and Pachal (1983) have extended Singh (1964) model on a probability distribution on the time of first birth in which it is assumed that a conception taking place during the infecundable period following the effective marriage with conception rate λ. As it is further noted that the fecundability parameter λ varies from the individual to individual even with same group. Biswas and Pachal (1988) have considered weighting of the Poission Process by Gamma distribution the renewal structure of the process is completely distroyed leading to inter-arrival distribution having infinitely divisible structure with dependent increments. In our study we have considered weighing of the Poisson process by weibull distribution and the results so obtained has been compared with the earlier existing results. On assuming that the fecundability parameter λ follows Weibull distribution with probability density function given by

$$\phi(t) = \frac{k}{a}\left[\frac{\lambda-\mu}{a}\right]^{k-1} e^{\left[\frac{\lambda-\mu}{a}\right]^{k}} \qquad a > 0,\ k > 0.$$

In particular $\mu = 0$, a=1

$$\phi(\lambda) = k\lambda^{k-1} \qquad 0 \le \lambda \le \infty$$

$$\psi(t) = \int_0^{\infty} f(t|\lambda)\ \phi(\lambda)\ d\lambda$$

$$= \int_0^{\infty} \frac{\lambda\ e^{-\lambda t}\ k\ (\frac{\lambda-\mu}{a})^{k-1}\ e^{-(\lambda-\mu)}\ d\lambda}{a}$$

Put $(\frac{\lambda-\mu}{a}) = u$

$$k(\frac{\lambda-\mu}{a})^{k-1}\ \frac{1}{a}\ d\lambda = du$$

$$= \int_0^{\infty} \lambda\ e^{-\lambda t}\ e^{-u}\ du$$

for u=0, a=1

$$= \int_0^{\infty} \lambda\ e^{-\lambda t}\ k\ \lambda^{k-1}\ e^{-\lambda}\ d\lambda$$

$$= \int_0^{\infty} k\ e^{-\lambda t}\ \lambda^{k}\ e^{-k}\ d\lambda$$

By considering the distribution for λ as Weibull the calculation becomes more complicated therefore we consider the distribution λ as a Standard Weibull distribution. Hence Weibull distribution is not desirable for the analysis in our study.

Feller (1957) showed taht the number of conceptions to a couple during time interval (o,t]

follows a Poisson Process with

$$P[\ N(t) = k] = \frac{e^{-\lambda t}\ (\lambda t)^k}{k!}$$

Where N(t) is the number of conceptions during the time interval (0,t].

The probability of no conception is $P[N(t)] = e^{-\lambda t}$ or $P[X \leq t] = F(t) = 1 - e^{\lambda t}$ fot given λ. The conditional waiting time distribution for the first conception is thus given by

$$f(t|\lambda) = \lambda\ e^{-\lambda t}\ ;\ 0 \leq t \leq \infty$$

The distribution is truncated at T' under this assumption of finite reproductive span for a woman is given by

$$f(t|\lambda) = \frac{\lambda e^{-\lambda t}}{\int_0^{T'} \lambda\ e^{-\lambda t}\ dt} = \frac{\lambda\ e^{-\lambda t}}{1 - e^{-\lambda t}}\ ;\ 0 \leq t < T'$$

and the unconditional distribution of the waiting time from marriage to first conception with the fecundable parameter λ following Pearsonian type III is given by

$$f(t) = k\ a^k \sum_{k=0}^{\infty} \frac{1}{(a+t+k\ T')^{k+1}}$$

$$\phi(\lambda) = a\ e^{-\lambda}\ ;\ \lambda \geq 0.$$

Hence the unconditional distribution of t is given by

$$\begin{aligned}
\psi(t) &= \int_0^\infty f(t|\lambda)\ \phi(\lambda)\ d\lambda \\
&= \int_0^\infty \lambda\ e^{-\lambda t}\ a\ e^{-\lambda t}\ d\lambda \\
&= a\lambda \int_0^\infty e^{-(t+a)\lambda}\ d\lambda \\
&= a \int_0^\infty \lambda^{2-1}\ e^{-(a+t)\lambda}\ d\lambda \\
&= \frac{a\ \Gamma(2)}{(a+t)^2} \\
&= \frac{a}{(a+t)^2} \qquad ;\ a > 0
\end{aligned}$$

However , there are several ways in which the model can be generalised. Let us take $h(t|\lambda) = \lambda\ e^{-\delta t}$, $\delta > 0$. instead of $h(t|\lambda) = \lambda$ based on which (4.5) is developed.

Let G(z,t) = Probability distribution function of $V_r(t)$ for all r = 1,2,...,n. $= \Sigma\ z^r\ V_r(t)$

$$= \sum_{r=0}^{\infty} z^r \int_0^n V_r(t|\lambda)\ \phi(\lambda)\ d\lambda$$

where $V_r(t) = \dfrac{(\lambda \int_0^t e^{-\delta t}\ dt)^r}{r!}\ e^{-\lambda \int_0^t e^{-\delta t} dt}$

$$= \frac{\lambda^r}{r!} \left[\frac{1-e^{-\delta t}}{\delta}\right]^r e^{-\lambda\left(\frac{1-e^{-\delta t}}{\delta}\right)}$$

$$= \frac{[\ \lambda\ A(t)\ ]^r}{r!}\ e^{-\lambda A(t)}$$

where $A(t) = \frac{1}{\delta}(1 - e^{-\delta t})$

Thus $G(z,t) = \sum_{r=0}^{\infty} z^r \frac{(\lambda A(t))^r}{r!} e^{-\lambda A(t)} a\, e^{-a\lambda}\, d\lambda$

$$= \sum_{r=0}^{\infty} \frac{z^r (A(t)^r \Gamma(r+1)}{r! (a+A(t))^{r+1}}$$

$$= \sum_{r=0}^{\infty} \frac{z^r (A(t))^r \Gamma(r+1)}{r! (a+A(t))^{r+1}}$$

$$= \sum_{r=0}^{\infty} \frac{z^r [A(t)]^r r!}{r! [a+A(t)]^{r+1}}$$

$$= \frac{1}{a+A(t)} \sum_{r=0}^{\infty} \left[\frac{Z A(t)}{a+A(t)} \right]^r$$

$$= \frac{1}{a+A(t)} \cdot \frac{1}{\left[1 - \frac{z A(t)}{a + A(t)}\right]}$$

$$= \frac{1}{a + A(t) - ZA(t)}$$

the choosing of the Pearsonian type III distribution for the fecundability parameter λ is that for suitable parameters the distribution is unimodel and has the value zero at the limits of its range zero and infinity. Hence it is reasonable to assume that most of the women in apopulation will have expectations near the average and that the number of each fertility level

will decrease to a small proportion of the total as the expectation of the child bearing fails to zero or increases to very high rates and thus rates and thus justifying the choice of type III function. The choice of this function is justified by its flexibility too.

The expected waiting time for the first conception is given by

$$E(t) = \left[\frac{a}{k-1} \left[1 + \frac{a}{a+t'} \right]^{k-1} \right]$$

Also we have

$$E(t^2) = \frac{a}{(k-1)(k-2)} \left[2a - (kt'-2a) \left(\frac{a}{a+t'} \right)^{k-1} \right.$$

and

$$Var(t) = \frac{a}{(k-1)^2} \left[\frac{ka}{(k-2)} - \left(\frac{a}{a+t'} \right)^{k-1} \right.$$

$$\left. \left(\frac{kt'(k-1)+2a}{(k-2)} + a\left[\frac{a}{a+t'}\right]^{k-1} \right) \right]$$

The maximum time exposure for the first conception from effective marriage is taken to be 5 years since we are concerned with the proportion of women who has their first birth during first five years of their marriage . 'Effective marriage ' in means that the wife was formally married, had begun living with her husband and had begun mensturating. Under the assumption that

reproductive span i.e 120 months the women who have not concieved within five years of her marriage will concieve in next five years of their marriage.

$$P_{j,j=+1} = \int_{j}^{j+1} f(t)\, dt$$

$$= a^k \sum_{k'=0}^{\infty} \left[\frac{1}{(a+j+kt')^k} - \frac{1}{(a+j+kt')^k} \right]$$

On substituting the value of a anf k will provide us the probability of the first conception for a married women at the above mentioned age group.

4.2 PROBABILITY MODEL FOR WAITING TIME OF SECOND CONCEPTION WHEN FIRST TOOK PLACE (AT T=0).:

If the hazard rate for conception is given by

$h(t) = \lambda\ e^{-\delta t}$ which represents the decreasing function for $\delta > 0$ and λ being the intensity function for conception.

For given λ, we have for time dependent poisson process

$$V_n(t|\lambda) = \frac{[\lambda \int_0^t e^{\delta t} dt]}{n!} \left[e^{- [\lambda \int_0^t e^{\delta t} dt]} \right]$$

$V_n(t)$ being the probability of occurance of n

conceptions or events in time (0,t].

Thus conditional probability of concieving upto time $\leq t$, given λ is as follows:

$$1 - V_0(t|\lambda) = 1 - e^{-\lambda \int_0^t e^{\delta t}\, dt}$$

$$= e^{-\lambda A(t)}$$

where $A(t) = \frac{1}{\delta}(1-e^{\delta t})$

λ varies from individual to individual following Pearsonian type III .

The probability that one conception takes place is

$$F_0(t) = \frac{d}{dt}[1 - V_0(t)]$$

$$f_0(t) = \frac{k\, a^k\, e^{\delta t}}{(a+A(t))^{k+1}}\, dt$$

$$\Rightarrow f(t) = \frac{k\, a^k\, e^{\delta t}}{(a+A(t)))^{k+1}} \cdot \frac{1}{1 - \left(\frac{\delta a}{1+\delta a}\right)^k}$$

Thus the expected waiting time from marriage to first conception is obtained from

$$E(t) = \int_0^\infty t\, f(t)\, dt$$

and t' represents the waiting time of the second conception given the first has occured at $t=0$.

$1 - \phi'(t)$ is the conditional probability of not concieving given first has occured at $t=0$.

Thus $\frac{d}{dt}\{ 1 - \phi'(t)\} = f'(t)$ which is conditional probability density function of waiting time of the second conception given the first has occured at time $t=0$.

$$= \frac{d}{dt}\left[1 - \frac{a^{k+1}\, A'(t)}{(a+A(t))^{k+1}} \right]$$

where $A'(t) = e^{\delta t}$

and $A(t) = \frac{1}{\delta}(1 - e^{\delta t})$

Thus the expected waiting time for the second conception given that one has occured at $t=0$ is given by

$$E(t' | t=0) = \int_0^\infty t\, f'(t)\, dt.$$

Since the reproductive wastage is less and the medical facilities are good thus the actual foetal wastage is less leading to certain interesting methodology describing the probability distribution of

the waiting time from the marrige to first conception and from first conception to second conception with time dependent increasing hazard rate for conception . Thus the present study will reveal that the expected waiting time from the marriage to first conception is longer than that from first to second.

CHAPTER - 5

A STERLIZATION POLICY USING MULTISTATE MARKOV CHAIN MODEL

5.1 The number of surviving children surviving is taken as the basis for Sterilization Policy for the reduction of the birth rate in a population. In general , a sterilization policy is implemented after a couple has a desired number of children on assuming a certain hypothetical survival rates operating in future. However, this method has no rationale to support : viz., the long term effect of mortality in deciding the number of surviving children may not be properly accounted for. An improved technique of assessing the long term impact of such sterilization policies is therefore to consider the joint effect of fertility and mortality which is the appropriate rationale for the present exercise. A Multistate Markov Chain model has been developed corresponding to varying fertility and mortality intensities at different levels (States) of surviving children of the couple. The asymptotic probabilities of having a fixed number of children have been derived.

5.2 DEVELOPMENT OF THE MODEL :

NOTATIONS:

(i) $P_n(t)$ = Probability of n number of children upto a time t, (n = 0.1.2.3.....)

(ii) $\lambda_n \Delta + O(\Delta)$ = Probability of having (n + 1) children in (t + Δ) given that n number of children are surviving at time t. (n = 0.1.2.3.....)

(iii) $M_n \Delta + O(\Delta)$ = Probability of having (n-1) children in (t + Δ) given that n number of children are surviving at time t, (n = 0.1.2.3.....)

(iv) M = A possible upper limit of the number of children to a couple.

(v) (M-K+1) = Number of surviving children at the time of sterilization based on certain policy. $1 \leq k \leq (M+1)$

$$P_0(t+\Delta) = P_0(t)\ (\ 1-\lambda_0\Delta + O(\Delta)) + P_1(t)(M_1\Delta + O\Delta)) \quad ..(5.1)$$

$$\begin{aligned} P_n(t+\Delta) = {} & P_n(t)[\ 1-(\lambda_n+M_n)\Delta + O(\Delta)] \\ & + P_{n-1}(t)[\lambda_{n-1}\Delta + O(\Delta)] \\ & + P_{n+1}(t)[M_{n+1}\Delta + O(\Delta)] \quad \ldots\ldots\ldots \quad (5.2) \end{aligned}$$

for $0 < n \leq M-k$

and

$$P_{M-k-1}(t+\Delta) = P_{M-k}(t)(\lambda_{M-k}\Delta + O(\Delta)) + P_{M-k+1}(t)(1-M_{M-k+1}\Delta+O(\Delta)) \ldots\ldots (5.3)$$

Now

$$(5.1) \Rightarrow P_o'(t) = -\lambda_0 P_o(t) + M_1 P_1(t) \ldots (5.4)$$

So on

$$(5.2) \Rightarrow P_n'(t) = -(\lambda_n + M_n) P_n(t) + P_{n-1}(t) \lambda_{n-1} + P_{n+1}(t) M_{n+1} \ldots\ldots (5.5)$$

for $0 < n \leq M-k$

and

$$(5.3) \Rightarrow P'_{M-k+1}(t) = -M_{M-k+1} P_{M-K+1}(t) + \lambda_{M-k} P_{M-k}(t) \ldots (5.6)$$

5.3 STEADY STATE SOLUTION :

Steady state solution of (5.4), (5.5) and (5.6) are obtainable by assuming

$P_0(\infty) = P_n(\infty) = P_{M-k+1}(\infty) = 0.$

Applying these conditions and solving successively the (M-K+2) equations, the parameters $\lambda_0, \lambda_1, \ldots \lambda_{M-k+1}, M_1, M_2, \ldots M_{M-k}$ parameters are obtained as follows :-

$$P_0(\infty) = \pi_0 = \left[1 + \frac{\lambda_0}{M_1} + \frac{\lambda_0\lambda_1}{M_1M_2} + \ldots + \frac{\lambda_0\lambda_1 \cdots \lambda_{M-(k+1)}}{M_1M_2 \cdots M_{M-k}} \right]^{-1}$$

$$P_n(\infty) = \frac{\lambda_0\lambda_1\cdots\cdots\lambda_{n-1}}{M_1M_2\cdots\cdots M_n}\ \pi_0$$

for all $o<n\leq M-k$

$$P_{M-k+1}(\infty) = \left[\ 1 + \frac{M_{M-k+1}}{\lambda_{M-k}} + \frac{M_{M-k+1}\ M_{M-k}}{\lambda_{M-k}\ \lambda_{M-(k+1)}} + \ldots + \frac{M_{M-k+1}M_{M-k}\cdots M_2M_1}{\lambda_{M-k}\ \lambda_{M-(k+1)}\cdots\lambda_0}\ \right]^{-1}$$

Examples :

Case I : For M-K-1 = 2.

i.e. sterilization is performed on attainment of two surviving children. Parameter matrix is $\begin{pmatrix}\lambda_0 & \lambda_1\\ M_1 & M_2\end{pmatrix}$

$$\pi_0 = \left[1+ \frac{\lambda_0}{M_1} + \frac{\lambda_0\lambda_1}{M_1M_2}\ \right]^{-1}$$

$$\pi_1 = \frac{\lambda_0}{M_1}\ \pi_0$$

$$\pi_2 = \left[\ 1+ \frac{M_2}{\lambda_1} + \frac{M_2M_1}{\lambda_1\lambda_0}\ \right]^{-1}$$

Case II : For M-K+1 = 3.

i.e. sterilization is performed on attainment of three surviving children. Parameter matrix is $\begin{pmatrix}\lambda_0 & \lambda_1 & \lambda_2\\ M_1 & M_2 & M_3\end{pmatrix}$

$$\pi_0 = [1+\lambda_0+\lambda_0\lambda_1+\lambda_0\lambda_1\lambda_2]^{-1}$$

$$\pi_1 = \lambda_0 \pi_0$$

$$\pi_2 = \lambda_0 \lambda_1 \pi_0$$

$$\pi_2 = [\ 1+ M_3 +M_3M_2+M_3M_2M_1]^{-1}$$

For presenting the effect of different levels of sterilization at various parity levels of the couples (from two to three surviving children), we define the fertility and mortality classes as follows:

FERTILITY AND MORTALITY <u>CLASS I</u>:

λ_0 =.20, λ_1 = .15, λ_2= .12

λ_3 =.10, λ_4 = .08, λ_5= .06;

M_1 = 0.05, $M_2=M_3=M_4=M_5$=0.04

FERTILITY AND MORTALITY <u>CLASS II</u>:

λ_0 =.30, λ_1 = .25, λ_2= .22

λ_3 =.20, λ_4 = .18, λ_5= .16;

M_1 = 0.06, $M_2=M_3=M_4=M_5$=0.05

Steady state probability distributions of the

number of surviving children classified by different levels of sterlization is given below :

Table 5.1

Number of Surviving Children	Levels of Sterlization at parties			
	2	3	4	5
0	0.05	0.02	0.01	0.002
1	0.20	0.06	0.02	0.01
2	0.75	0.23	0.08	0.04
3		0.69	0.25	0.11
4			0.63	0.28
5				0.56

Table 5.2

Number of Surviving Children	Levels of Sterlization at parties			
	2	3	4	5
0	0.03	0.01	0.002	0.001
1	0.16	0.04	0.01	0.002
2	0.81	0.18	0.04	0.01
3		0.78	0.19	0.05
4			0.76	0.20
5				0.73

5.5 DISCUSSION :

Proportion of surviving children at all levels of fertility and mortality classes under consideration decreases consistently when sterilization levels are relaxed. The proportion of couples having two surviving children in the long run is 75% when sterilization is made on attainment of two surviving children as against only 56% of the couples having five surviving children when sterilization level is relaxed to the extent of sterilizing couples only after having five surviving children.

The same trend is more or less revealed in table corresponding to different sets of fertility and mortality levels as may immediately be seen on inspection of the diagonal elements of the matrix corresponding to the table. However, a better insight into the variation of the proportion of surviving children as a result of changes in the level of parity under consideration for adopting a sterilization policy may be obtained when we consider the average no of surviving children under different F.M. set up. This is considered in the following table.

Table 5.3

Mean Number of Surviving children at different levels of Sterlization

F.M. Class	Level of Sterlization at number of Surviving Children			
	2	3	4	5
I	1.70	2.60	3.49	4.33
II	1.77	2.73	3.69	4.65

Table 5.2 provides a clue for deciding the optimal level of sterilization depending on the mean num ber of surviving children. For example, if the policy is to ensure at least one surving male child in the long run (as it is most customary in India) per couple then it may be seen that irrespective of the variations in the levels of the fertility and mortality the level of sterilization may optimally be made on the basis of three surviving children. Whereas if sterilization policy is relaxed to the extent of sterilizing couples only on the basis of more than three number of surviving children then one may expect a phenomenal increase in the growth rate of the population not

consistent with the situation leading to adopt a sterilization policy motivated to cut down the growth rate of the population. On the other hand if sterilization policy is made on the basis of only two surviving children then the condition of ensuring at least one male child in the long run per couple may not be realized although the same may fulfil the plan of reducing the growth rate in a fast increasing population. Such a policy may not be culturally and traditionally accepted by the people at large, thus the average number of children per family with one male child is three.

5.6 MARTINGALE APPROACH :(STERILIZATION POLICY)

INTRODUCTION:

For a problem of sterilization of mothers with i_o number of suriviving children, attained by addition of a birth at the starting point of the observation, Assuming these mothers may be considered homogeneous with respect to age, i_o is a minimum number of children after which the mother is eligible for sterlization.

However, if i_o is reduced to $(i_o - 1)$ then the mother concerned, is discarded for sterilization once

for all. Whereas those mothers whose numnber of surviving children move from i_o to $(i_o + 1)$ first (without passing through any other state) is considered eligible for sterilization and they are sterilized at a random time S_T on attainment of $(i_o + 1)$ number of surviving children. Besides these two groups, the proportion of women who will remain in state S_i, number of surviving children throughout a period T from the starting point of the observation (the reference period π which is considered large enough to cover all the three types of outcomes) will be discarded from the sterilization programme.

The asymptotic estimator (for large T) of the above have been attempted to be obtained using Martingale stopping rule.

5.7DEVELOPMENT OF THE MODEL:

NOTATIONS:

1) X(t) is the Number of surviving children at any time t of the birth and death process with birth and death parameters $\lambda_i = \lambda(i) = \lambda$; $M_i = M(i) = M$ respectively.

2) $\pi_0 = P[x(t) = i_0 - 1 \mid X(0) = i_0$;
$X(j) = i_o$
for all $j<t]$

where π_0 is the asymptotic probability.

3) $\alpha(T) = P[X(j) = i_0 | X(0) = i_0$ for all $j \epsilon (0,T)]$.

4) $P_k(t) = P[X(t) = k]$; $k \geq 0$.

We have

$$P_k(t+\Delta) = P_k(t)(1-(\lambda_k \Delta + M_k \Delta) + P_{k-1}(t)\lambda_{k-1}\Delta + P_{k+1}(t)M_{k+1}\Delta + o(\Delta)) \quad \text{for all } k \geq 0 \quad \ldots(5.8)$$

Subject to $P_{k-1}(t) = 0$; for $k=0$.

Let us define $f(j) = 1 + \frac{M_1}{\lambda_1} + \frac{M_1 M_2}{\lambda_1 \lambda_2} + \ldots + \frac{M_1 M_2 + \ldots M_{j-1}}{\lambda_1 \lambda_2 + \ldots + \lambda_{j-1}}$ for all $j>0$ ...(5.9)

Corresponding to the parameters $\lambda_i = \lambda(i)$ and $M_i = M(i)$ for all $i>0$

Then it can immediately be stated $F(X(t))$ is a Martingale with respect to $f_t = f(X(t))$ (a σ field) and $f_u = f(X(u)$; $0 \leq uP \leq t$ (Karlin and Taylor (1975)).

Let a stopping time be S_T

$$S_T = \min(t \geq 0 \; X(t) = i_0 - 1; X(t) = i_0 + 1 | X(0) = i_0) \quad \text{where } T = S_T \quad \ldots (5.10)$$

Where S_T represents the stopping time when $X(t) = i_0 - 1$, or $X(t) = i_0 + 1$ or $X(t) = i_0$ for all $(0,T]$. In this case $S_T = T$, and constant time T is a Markov time which are considered as absorbing states of the process given $X(0) = i_0$. Then by optional sampling theorem of Martingale (Karlin and Taylor (1975)).

$$E(f(X(S_T))) = E(f(X(0)) = E(f(i_0)) = f(i_0), \quad \ldots(5.11)$$

Since $X(0) = i_0$, a fixed number by assumption.

Then

$$E(f(i_0)) = f(i_0) = E(f(X_T))$$

$$= (1-\pi_0)[\alpha f(i_0)+(1-\alpha)f(i_0+1)] + \pi_0(f(i_0-1))$$

$$\Rightarrow \pi_0 = \frac{F(i_0) - (1-\alpha)f(i_0+1) - \alpha f(i_0)}{f(i_0-1) - (1-\alpha)f(i_0+1) - \alpha f(i_0)} \quad \ldots\ldots\ldots(5.12)$$

Then π_0 proportion will not be sterilized and $(1-\alpha)$ $(1-\pi_0)$ will be sterilized where

$\alpha = e^{-(\lambda_i - M_i)T}$ for large T ..... (5.13)

the reference period in the sterilization programme.

PARTICULAR CASE:

Let $i_0 = 2$

$$f(i_0-1) = 1 + \frac{M_1}{\lambda_1}$$

$$f(i_0) = 1 + \frac{M_1}{\lambda_1} + \frac{M_1 M_2}{\lambda_1 \lambda_2}$$

$$\pi_0 = \frac{\left[1+\frac{M_1}{\lambda_0}+\frac{M_1 M_2}{\lambda_0 \lambda_1}\right] - \left[(1-e^{(\lambda_2 - M_2)T}\right]\left[1+\frac{M_1}{\lambda_0}+\frac{M_1}{\lambda_0}+\frac{M_2}{\lambda_1}+ -e^{(\lambda_3 - M_3)T}\left[1+\frac{M_1}{\lambda_0}+\frac{M_1 M_2}{\lambda_0 \lambda_1}\right]\right.}{\left\{\left[1+\frac{M_1}{\lambda_0}\right] -(1-e^{-(\lambda_2-M)T}_2)\left(1+\frac{M_1}{\lambda_0}+\frac{M_1}{\lambda_0}+\frac{M_2}{\lambda_1}\right) -e^{-(\lambda_3-M)T}_3 \left(1+\frac{M_1}{\lambda_0}+\frac{M_1}{\lambda_0}+\frac{M_2}{\lambda_1}\right)\right\}} \quad \ldots (5.14)$$

Therefore, the total proportion of mothers not to be sterilized $=[\alpha(1-\pi_0) +\pi_0]$

where $\alpha = e^{-(\lambda - M)T}$ for given T ...(5.15)

Next, to obtain the expected stopping time S_T which decides the expected time when policy decision is taken for sterilization, $100(1-\alpha)(1-\pi_0)\%$ of mothers while leaving $100[\alpha(1-\pi_0)+\pi_0]$ % mothers for sterilization programme. We have

$$Y(t) = g(X(t))) - \lambda(X(u))(g(X(u)+) - G(X|u)) - M(X(u))(G(X(u))-g(X(u)-1))du \quad \ldots (5.16)$$

is a Martingale over $f_t = f(X(t))$ for random function g

where the expectation of Y(t) exists. Further, if g (i) be so chosen, so that

$$\lambda(i)[g(i+1)]-M(i)[g(i)-g(i-1)] = 1 \quad \ldots\ldots \quad (5.17)$$

for all i = 1,2,3..

then $Y(t) = (g(X(t))-t)$ is a Martingale

$\Rightarrow E(Y(S_T)) = E(Y(0)) = E[g(X(t))] - E(S_T 0$ ..(5.18) by Martingale optional sampling theorem.

Further, $Y(0) = g(X(0)) = g(i_0)$

$$= g(i_0) = E(g(X(S_T))) - E(S_T) \ldots (5.19)$$

$$= E(g(X(S_T))) - g(i_0) = E(S_T).$$

To obtain $E(S_T)$, we require the solution of the difference equation given by (5.17).

If $g(i+1)-g(i) = \Delta g(i) = V(i)$.

We have the transfdormed equation of (5.17) as

$$V(i) = (\lambda(i)V(i) - M(i)V(i)) = 1.$$

We may immediately get

$$V(0) = 0, V(1) = (g(2) - g(1)) = \frac{1}{\lambda_1}$$

$$V(2) = (g(3) - g(3)) = \frac{1}{\lambda_2} \quad [1 + \frac{M_2}{\lambda_1}]$$

and in general

$$V(i) = \frac{1}{\lambda_i} \left[1 + \frac{M_{i-1}}{\lambda_{i-2}} \left(1 + \frac{M_{i-1}}{\lambda_{i-2}} \right)(..) \left(1 + \frac{M_2}{\lambda_1}\right) \right] \quad \ldots\ldots(5.20)$$

for $i \geq 0$

This gives the solution g(i) of (5.17) as

$$g(i) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \left(1 + \frac{M_2}{\lambda_1} + \ldots. \right) \left(1 + \frac{M_3}{\lambda_2} \left(1 + \frac{M_2}{\lambda_1} \right) \right)$$

for $i \geq 0$

and $g(1) = 0$, $g(0) = 0$ etc.

In particular, for $i_0 = 3$, $g(i_0 - 1) = \frac{1}{\lambda_1}$

$$g(3) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \left(1 + \frac{M_2}{\lambda_1} \right)$$

$$E(g(X(S_T))) = (1-\pi_0)(1-\alpha)\, g(i_0+1) + \alpha(1-\pi_0) g(i_0) + \pi_0 g(i_0+1).$$

for $i_0 = 3$

$E(S_T)\ E(g(X(S_T))) - g(i_0)$

Where f, g and α are given in (5.9), (5.15) and (5.17) respectively.

5.9 CONCLUSION:

Thus by taking any numerical Illustration it can be seen that for $i_o = 2$ (different values of M_i, λ_i) may be the suitable most starting numnber of surviving

children in this Sterilization Programme under the hypothetical values of the parameters. Thus this decides the policy of the sterilization programme.

CONCLUSION

This Thesis pertains to the Analysis of some Stochastic Models and its applications in the area of fertility analysis and family planning programmes.

The study is initiated with a technique based on Palm Probbability for the comparision of fertility statuses of two groups which are identical on Socio-economic parameters and different on age of effective marriage. The idea is to examine the policy of increase of conception by postponement of marriage. Thus it is believed that the methodology developed may be tested on a more exhaustive set of real data to derive more meaningful result.

The model building exercise has been modified here by using Palm Probabilistic technique with restriction of finite marital exposure. Thus comparision of results based upon the approach of infinite marital exposure with that of finite marital exposure(Shreshta and Biswas 1985) reveals that the gain in precision in the estimate of interconception intervals for higher order of concptions. However for the higher order of conceptions further investigations are necessary in this case.

In chapter four the expressions for the expected waiting time for the r^{th} live birth for all r and its variance were derived and a method to obtain the estimates of the parameters by the method of moments. Further the same method can be generalised by assuming p the probability of having a conception following a ß distribution in confirmity with Potter and Parker(1964), which no doubt complicates the estimational techniques. But the complications remain worth adopting if we look into the practical utility of the same. Similarly fecundability parameter λ may be assumed to vary among inidividuals following gamma distribution (with two parameters following BRASS (1958)) in which case the unconditional distribution of the waiting time of the first conception would be a Pareto distribution which will again complicate the estimation technique, but again complication are seen in taking Weibull distribution. It has been noted that the estimation by the method of moments from Pareto distribution may be considerably simplified by taking the correlation of interconceptive delays in a hetrogenious population. Another notable feature in the finding of the study is that when comparison is made between the estimates of the monthly probability of

conceptions using the probability models both with time independent hazard rate with those obtained with time dependent hazard, the probability values seem to be on the higher side in the latter case. A proportion of pregnancy wastage is less and stress due to lack of sex education leads to under estimate of waiting time of conceptions and finally leading to the over estimate of monthly probabilities of conceptions. While developing the probability models for the waiting time distribution for different order of conceptions we have assumed that the parameters remain the same from parity to parity for the same individual i.e. for different order of conceptions the fecundability parameter follows the same distribution with the same pair of parameters a and k. This is leaving the situation in a dependent process where Palm probability technique has been made to obtain the waiting time distribution for the successive order of conceptions given that the previous order of conception has taken place at a given time T=0. Thus the effective marriage means that the wife was formally married and had begun living with her spouse.

In Chapter-5 attempts have been made to develop Multisate Markov Chain model based on a density

dependent birth and death process to obtain the asymptotic probability of having fixed number of surviving children based upon the same distribution. Attempts have also been made to decide an optimal sterlization policy depending upon the number of surviving children. If the policy is to ensure atleast one surviving male child in the long run then the results shows that the irrespective of the variations in the levels of fertility and mortality the levels of sterlization may be made on the basis of three surviving children. If the policy is relaxed to the extent of sterlising mother s only on the basis of less than three or more than three surviving children with atleast one male child , then one may expect the under or over growth rate which may not be consistent with the policy of reduction of growth rate. Finally, if an intermediate action based on the policy of three surviving children is adopted then the condition of ensuring one surviving male child in the long run per couple may only be realised with low probability. Martingales have been constructed on the continuous parameters , birth and death process to obtain the stopping time for sterlizing mother, for sterlization in the future, given that each of them have a fixed

number of children. This methodology is useful in answering two basic questions 1- the cost of the sterlization programme and 2- stopping time. The percentage of mother's obtained for the sterlization programme based on the choice of different number of surviving children is indicative of the increase or decrease of the probable cost if we change over a more liberal policy while adopting the sterlization programme. Thus one can indirectly calculate the age of mother at the time of sterlization connected with the various sterlization programme. These two problems may be of great importance and should be kept in view in fram ing of the national policy programme.

However, since the basic objective of this study is toestimate the correlation between waiting time for the first and second order of conception and its relation with the age of women and the application of Palm probability, Counter Model, Renewal theory and other Stochastic processes oriented methodologies as Martingale theory are some of the applications for arriving at the solution of fertility analysis which are methodological in nature and solutions are derived at theoritical plane and their justification depends on more exhaustive real data.

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